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# Eleven dimensional supergravity in light cone gauge

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**Abstract**

Light-cone gauge manifestly supersymmetric formulation of eleven dimensional supergravity is developed. The formulation is given entirely in terms of light cone scalar superfield, allowing us to treat all component fields on an equal footing. All higher derivative on mass shell manifestly supersymmetric 4-point functions invariant with respect to linear supersymmetry transformations and corresponding (in gravitational bosonic sector) to terms constructed from four Riemann tensors and derivatives are found. Superspace representation for 4-point scattering amplitudes is also obtained. Superfield representation of linearized interaction vertex of superparticle and supergravity fields is presented. All 4-point higher derivative interaction vertices of ten-dimensional supersymmetric Yang-Mills theory are also determined.

# 1 Introduction and Summary

## 1.1 Motivations for light-cone gauge approach

In view of high degree of symmetry $11d$ supergravity [1] has attracted considerable interest for a long period of time. Presently due to conjectured interrelation with superstrings interest in $11d$ supergravity is renewed and by now $11d$ supergravity is viewed as low energy approximation of M-theory[2, 3]. Because application of the light cone approach turned out to be fruitful in many problems of superstrings one can expect this approach might also be useful to understand M-theory better. Extensive studies triggered by a conjecture made in Ref.[4] support this expectation. Therefore one can believe that study of various aspects of $11d$ supergravity in the framework of light cone approach is a fruitful direction to go. This is what we are doing in this paper.

So far superfield light cone formalism was explored for $d \leq 10$ supergravity theories (see *e.g.* [5, 6, 7, 8]). The major goal of this paper is to develop manifest supersymmetric light cone gauge formulation of $11d$ supergravity and discuss its various applications. Our method is conceptually very close to that used in [5]-[9] to find ten dimensional supersymmetric theories and is based essentially on a light cone gauge description of interaction vertices developed in [10] (see also [11, 12, 13]).

One of motivations of our investigation is to demonstrate efficiency of light cone formulation in study of various higher derivative 4-points functions which consist in their gravitational bosonic sector $\partial^{2n} R^4$ terms, where $R$ stands for Riemann tensor. We shall show that light cone formulation gives simple way to derive manifestly supersymmetric expressions for these vertices. As by-product of this study we find superspace representation for 4-point scattering amplitude of the generic $11d$ supergravity [1]. An attractive feature of our approach is that the methods we use are algebraic in nature and this allows us to extend our result to other supersymmetric theories in a rather straightforward way. That is to say that we shall extend our discussion to the case of $10d$, non-Abelian supersymmetric Yang-Mills (SYM) theory and we shall find manifestly supersymmetric representation of all 4-point functions, which consist in their bosonic sector $\partial^n F^4$ terms.

As illustration of our approach we shall begin with study of light cone gauge cubic interaction vertex of generic $11d$ supergravity. Our interest in light cone gauge re-formulation of the $11d$ supergravity cubic interaction vertex is motivated by the following reason. For the case of ten dimensional theories it is light cone gauge cubic vertices of $10d$ supergravity theories [5] that admit simple and natural extension to superstring theories [6]. Therefore it is reasonable to expect that it is our light cone gauge cubic interaction vertex of $11d$ supergravity that will have natural extension to M-theory, i.e. we expect that formulation of M-theory, which is still to be understood, should be simpler within the framework of light cone approach. The situation here may be analogous to that in string theory: a covariant formulation of closed string field theories is non-polynomial, while light cone formulation restricts the string action to cubic order in string fields.

Another application of our approach is a derivation of linearized interaction vertex of superparticle with $11d$ supergravity fields. As is well known in many cases an evaluation of scattering amplitudes taken to be in the form of quantum mechanical correlation function of world line linearized interaction vertices turns out to be more convenient as compared to evaluation in field theoretical approach. Therefore it seems highly likely that an extension of world line approach to the $11d$ supergravity should be fruitful from the perspective of future

applications to M-theory.

### 1.2 3-point and 4-point interaction vertices

To discuss light cone superspace formulation of $11d$ supergravity we use a superfield $\Phi(p,\lambda)$ which depends on bosonic momenta $p^I$, $p^+$, Grassmann momentum $\lambda$, while a dependence of the superfield $\Phi(p,\lambda)$ on the light cone evolution parameter $x^+$ is implicit [1]. Light cone gauge action describing dynamics of $11d$ supergravity fields admits then the following standard representation:

$$S = \int dx^+ d^{10}p\, d^8\lambda\ \Phi(-p,-\lambda) \mathrm{i}\, \beta \partial^- \Phi(p,\lambda) + \int dx^+ P^- \,, \tag{1.1}$$

where $P^-$ is the Hamiltonian and $\partial^- \equiv /\partial x^+$. In theories of interacting fields the Hamiltonian receives corrections having higher powers of physical fields and one has the following expansion

$$P^- = \sum_{n=2}^{\infty} P^-_{(n)} \,, \tag{1.2}$$

where $P^-_{(n)}$ stands for $n$ - point contribution (degree $n$ in physical fields) to the Hamiltonian. Dynamics of free fields is described by the well known free Hamiltonian $P^-_{(2)}$ to be discussed in Section 2, while $n$-point interaction corrections $P^-_{(n)}$ , $n \geq 3$, admit the following representation

$$P^-_{(n)} = \int d\Gamma_n \prod_{a=1}^{n} \Phi(p_a, \lambda_a)\, p^-_{(n)} \,. \tag{1.3}$$

Density $p^-_{(n)}$, sometimes referred to as $n$- point interaction vertex, depends on light cone momenta $\beta_a$, transverse momenta $p_a^I$ and Grassmann momenta $\lambda_a$, where an external line index $a = 1, \ldots, n$ labels $n$ interacting fields. Explicit expression for an integration measure $d\Gamma_n$ is given below in (3.7)-(3.9).

We begin with discussion of cubic interaction vertex for generic $11d$ supergravity theory [1]. Expression for the 3-point interaction vertex we find takes the following form:

$$\begin{aligned} \frac{3}{\kappa} p^-_{(3)} \ &= \ \mathbb{P}^{L2} - \frac{\mathbb{P}^L}{2\sqrt{2}\hat{\beta}} \Lambda \not{\mathbb{P}} \Lambda + \frac{1}{16\hat{\beta}^2} (\Lambda \not{\mathbb{P}} \Lambda)^2 - \frac{|\mathbb{P}|^2}{9 \cdot 16 \hat{\beta}^2} (\Lambda \gamma^j \Lambda)^2 \\ &+ \ \frac{\mathbb{P}^R}{9 \cdot 16\sqrt{2}\hat{\beta}^3} \Lambda \not{\mathbb{P}} \Lambda (\Lambda \gamma^j \Lambda)^2 + \frac{\mathbb{P}^{R2}}{2^7 \cdot 63 \hat{\beta}^4} ((\Lambda \gamma^i \Lambda)^2)^2 \,, \end{aligned} \tag{1.4}$$

where we use the notation

$$\mathbb{P}^I = \frac{1}{3} \sum_{a=1}^{3} \check{\beta}_a p_a^I \,, \qquad \Lambda = \frac{1}{3} \sum_{a=1}^{3} \check{\beta}_a \lambda_a \,, \tag{1.5}$$

$$\check{\beta}_a \equiv \beta_{a+1} - \beta_{a+2} \,, \qquad \hat{\beta} \equiv \beta_1 \beta_2 \beta_3 \,, \tag{1.6}$$

[1]We decompose momenta $p^I$, $I = 1, \ldots, 9$ into $p^i$, $i = 1, \ldots, 7$ and $p^{R,L}$ where $p^{R,L} \equiv (p^8 \pm \mathrm{i} p^9)/\sqrt{2}$. Grassmann momentum $\lambda$, whose spinor indices are implicit, transforms in spin one-half representation of $so(7)$. For momentum in light-cone direction we use simplified notation $\beta \equiv p^+$.

$\not{\mathbb{P}} \equiv \gamma^i \mathbb{P}^i$, $|\mathbb{P}|^2 \equiv \mathbb{P}^I \mathbb{P}^I$, $\gamma^i$ are $so(7)$ Dirac matrices, and we use identification $\beta_a \equiv \beta_{a+3}$ for 3-point interaction vertices. In formula (1.4) $\kappa$ is gravitational constant (see formula (4.39) for normalization). Note that cubic vertices for generic $11d$ supergravity are the only vertices which can be constructed in cubic approximation. In Section 4 we demonstrate that the Poincaré supersymmetries forbid supersymmetric extension of $R^2$ and $R^3$ terms.

We proceed to discussion of 4-point interaction vertices. It should be emphasized from the very beginning that we do not consider 4-point interaction vertices of the generic $11d$ supergravity [1]. We find 4-point vertices that are invariant with respect to linear supersymmetry transformations and do not depend on contributions of exchanges generated by cubic vertices. These 4-point vertices in their gravitational bosonic sector involve higher derivative terms that can be constructed from Riemann tensor and derivatives, i.e. they can be presented schematically as $\partial^{2n} R^4$. The bosonic bodies of these supersymmetric vertices appeared in various previous studies and they are of interest because they are responsible for quantum corrections to classical action of $11d$ supergravity.

4-point interaction vertices $p_{(4)}^-$ depend on momenta $p_a^I$ and $\lambda_a$ through the following quantities

$$\mathbb{P}_{ab}^I \equiv p_a^I \beta_b - p_b^I \beta_a \,, \qquad \Lambda_{ab} \equiv \lambda_a \beta_b - \lambda_b \beta_a \,, \tag{1.7}$$

where external line indices $a, b$ take values $a, b = 1, 2, 3, 4$. Solution to 4-point interaction vertices we find admits the representation

$$p_{(4)}^- = -\Big( (\mathbf{J}_{12}\mathbf{J}_{34})^2 ut E_s + (\mathbf{J}_{13}\mathbf{J}_{24})^2 st E_u + (\mathbf{J}_{14}\mathbf{J}_{23})^2 us E_t \Big) g(s,t,u) \,, \tag{1.8}$$

where we use the notation

$$\mathbf{J}_{ab} \equiv \mathbb{P}_{ab}^L \exp\Big( \frac{\Lambda_{ab} \not{q}_{ab} \Lambda_{ab}}{4\sqrt{2}\beta_a \beta_b \beta_{ab}} \Big) \,, \tag{1.9}$$

$$E_u \equiv \exp\Big( - \frac{u \Lambda^L \not{q}_{{}_L} \Lambda^L}{2\sqrt{2}\beta_{13} q_{{}_L}^2 (\mathbb{P}_{13}^L \mathbb{P}_{24}^L)^2} \Big) \,, \tag{1.10}$$

$$q_{ab}^i \equiv \frac{\mathbb{P}_{ab}^i}{\mathbb{P}_{ab}^L} \,, \qquad \beta_{ab} \equiv \beta_a + \beta_b \,, \tag{1.11}$$

$$q_{{}_L}^i \equiv q_{13}^i - q_{24}^i \,, \qquad \Lambda^L \equiv \Lambda_{13} \mathbb{P}_{24}^L - \Lambda_{24} \mathbb{P}_{13}^L \,, \tag{1.12}$$

$\not{q} \equiv \gamma^i q^i$, $q_{{}_L}^2 \equiv q_{{}_L}^i q_{{}_L}^i$. In above-given expressions the quantities $s$, $t$, $u$ are the standard Mandelstam variables (normalization we use for these variables may be found in (5.14),(5.16)), while the quantities $E_s$ and $E_t$ are obtainable from $E_u$ by appropriate interchange of external line indices 1,2,3,4:

$$E_s \equiv E_u|_{2\leftrightarrow 3} \,, \qquad E_t \equiv E_u|_{3\leftrightarrow 4} \,. \tag{1.13}$$

Explicit form of a function $g(s,t,u)$ (1.8), which should be symmetric in Mandelstam variables, cannot be fixed by exploiting restrictions imposed by global symmetries alone. This function is freedom of our solution. Assuming that $g(s,t,u)$ admits Taylor series expansion we get the following lower order terms in infinite series expansion of $g(s,t,u)$:

$$\begin{aligned} g(s,t,u) &= g_0 + g_2(s^2+t^2+u^2) + g_3 stu + g_4(s^4+t^4+u^4) \\ &+ g_5 stu(s^2+t^2+u^2) + g_{6;1}(s^6+t^6+u^6) + g_{6;2}(stu)^2 + \ldots \,. \end{aligned} \tag{1.14}$$

Sign minus in r.h.s. of formula (1.8) is of no physical significance and is chosen for later convenience.

Our approach allows us to obtain surprisingly compact superspace representation for 4-point scattering amplitude of generic $11d$ supergravity theory [1] which we exhibit here:

$$\mathcal{A}_{(4)} = 2\kappa^2\Big(\frac{(\mathbf{J}_{12}\mathbf{J}_{34})^2}{s}E_s + \frac{(\mathbf{J}_{13}\mathbf{J}_{24})^2}{u}E_u + \frac{(\mathbf{J}_{14}\mathbf{J}_{23})^2}{t}E_t\Big)\,. \tag{1.15}$$

### 1.3 Contents of the rest of the paper

The rest of the paper contains derivation of above-mentioned 3-point and 4-point interaction vertices, superfield description of linearized interaction of free superparticle with $11d$ supergravity and related explanations and technical details. The paper is organized as follows.

In section 2 we introduce a notation and describe the free level $so(7)$ covariant formulation of $11d$ supergravity in terms of unconstrained light cone scalar superfield.

In section 3 we discuss arbitrary $n$-point interaction vertices and find constraints for these vertices imposed by symmetries of the Poincaré superalgebra.

In section 4 we study a cubic interaction vertex of $11d$ supergravity and find manifest supersymmetric light cone representation for this vertex. To do that we use method of [10], which allows us to find simple and compact representation for the vertex in question. Because this vertex describes generic $11d$ supergravity it involves terms having the second power of derivatives. The formalism we use is algebraic in nature and this allows us to study on an equal footing the vertices involving arbitrary powers of derivatives and describing in their gravitational bosonic sector $R^2$ and $R^3$ terms. We demonstrate explicitly that the Poincaré supersymmetries forbid supersymmetric extension of $R^2$ and $R^3$ terms.

In section 5 we study 4-point vertices that are invariant with respect to linear Poincaré supersymmetry transformations and involve arbitrary powers of derivatives. These 4-point supersymmetric vertices consist in their gravitational bosonic sectors $\partial^{2n}R^4$ terms. We find all constraints imposed by Poincaré supersymmetries on such vertices and find all possible solutions to these constraints. We present explicit and simple form for these 4-point vertices. Also, as a by-product of our investigation we present superspace representation for 4-point scattering amplitude of the generic $11d$ supergravity.

In section 6 we develop world line representation for interaction vertex of $11d$ supergravity. Namely, we obtain superfield description of linearized interaction of free superparticle with $11d$ supergravity.

In section 7 we extend out formalism to discuss 4-point vertices for $10d$ nonabelian SYM theory. Because our formalism is algebraic in nature it allows us, starting with above mentioned 4-point vertices of $11d$ supergravity, to write down in a straightforward way all higher derivatives 4-point interaction vertices for SYM theory. In their bosonic sector these supersymmetric vertices correspond to $\partial^n F^4$ terms.

Section 8 summarizes our conclusions and suggests directions for future research. Appendices contain some mathematical details and useful formulae.

## 2 Free $11d$ supergravity in $so(7)$ light cone basis

Method suggested in Ref.[14] reduces the problem of finding a new (light cone gauge) dynamical system to the problem of finding a new solution of commutation relations of an

defining symmetry algebra[2]. Because in our case the defining symmetries are generated by $11d$ Poincaré superalgebra we begin our investigation with description of the $so(7)$ form of this superalgebra, which is most convenient for our purposes. The conventional light cone formalism in eleven dimension based on $so(9)$ symmetries requires complicated superfield constraints which we prefer to avoid. Fortunately it turns out that reducing manifest symmetries to the $so(7)$ symmetries allows one to develop superfield light cone formulation of $11d$ supergravity in terms of unconstrained scalar superfield. Another reason why do we prefer to use the $so(7)$ light cone formulation is that it is the $so(7)$ symmetries that are manifest symmetries of the general method of constructing cubic interaction vertices developed in [10] (see section 4). In this section we focus on free fields.

Poincaré superalgebra of $11d$ Minkowski spacetime consists of translation generators $P^\mu$, rotation generators $J^{\mu\nu}$, which span $so(10,1)$ Lorentz algebra, and 32 Majorana supercharges $Q$. The Lorentz covariant form of (anti)commutation relations is

$$[P^\mu,\, J^{\nu\rho}] = \eta^{\mu\nu} P^\rho - \eta^{\mu\rho} P^\nu\,, \qquad [J^{\mu\sigma},\, J^{\nu\rho}] = \eta^{\sigma\nu} J^{\mu\rho} + 3 \text{ terms}, \tag{2.1}$$

$$[J^{\mu\nu},\, Q] = -\tfrac{1}{2}\gamma_{32}^{\mu\nu} Q\,, \qquad \{Q,\, Q\} = -\gamma_{32}^\mu C_{32}^{-1} P_\mu\,, \tag{2.2}$$

where $\gamma_{32}^\mu$ are $so(10,1)$ Dirac matrices and we use mostly positive flat metric tensor $\eta^{\mu\nu}$. The generators $P^\mu$ are chosen to be hermitian, while the $J^{\mu\nu}$ to be antihermitian. The supercharges $Q$ satisfy Majorana condition $Q^\dagger\gamma_{32}^0 = Q^t C_{32}$. To develop light cone formulation we introduce instead of the Lorentz basis coordinates $x^\mu$ the light cone basis coordinates $x^\pm$, $x^R$, $x^L$, $x^i$ defined by[3]

$$x^\pm \equiv \frac{1}{\sqrt{2}}(x^{10} \pm x^0), \qquad x^R \equiv \frac{1}{\sqrt{2}}(x^8 + \mathrm{i}x^9), \qquad x^L \equiv \frac{1}{\sqrt{2}}(x^8 - \mathrm{i}x^9) \tag{2.3}$$

and treat $x^+$ as an evolution parameter. In this notation Lorentz basis $11d$ vector $X^\mu$ is decomposed as $(X^+, X^-, X^I)$, where $X^I = (X^R, X^L, X^i)$. A scalar product of two $11d$ vectors is decomposed then as

$$\eta_{\mu\nu} X^\mu Y^\nu = X^+Y^- + X^-Y^+ + X^IY^I\,, \qquad X^IY^I = X^iY^i + X^RY^L + X^LY^R\,, \tag{2.4}$$

where the covariant and contravariant components of vectors are related as $X^+ = X_-$, $X^R = X_L = (X^L)^*$. In the light cone formalism Poincaré superalgebra splits into generators

$$P^+,\quad P^I,\quad J^{+I},\quad Q^{+R},\quad Q^{+L},\quad J^{+-},\quad J^{IJ}, \tag{2.5}$$

which we refer to as kinematical generators and

$$P^-,\quad J^{-I},\quad Q^{-R},\quad Q^{-L}\,, \tag{2.6}$$

which we refer to as dynamical generators. For $x^+ = 0$ the kinematical generators in the superfield realization are quadratic in the physical fields[4], while the dynamical generators receive higher-order interaction-dependent corrections.

[2]This method is Hamiltonian version of the Noether method of finding new dynamical system. Interesting recent discussion of the Noether method may be found in [15].

[3]$\mu, \nu = 0, 1, \ldots 10$ are $so(10,1)$ vector indices, $\alpha = 1, \ldots, 8$ is $so(7)$ spinor index, $I, J, K = 1, \ldots 9$ are $so(9)$ vector indices, $i, j, k = 1, \ldots, 7$ are $so(7)$ vector indices.

[4]Namely, for $x^+ \neq 0$ they have a structure $G = G_1 + x^+ G_2$, where $G_1$ is quadratic in fields, while $G_2$ contains higher order terms in fields.

The $so(7)$ form of Poincaré algebra commutators can be obtained from (2.1) by using the light cone metric having the following non vanishing elements $\eta^{+-} = \eta^{-+} = 1$, $\eta^{RL} = 1$, $\eta^{ij} = \delta^{ij}$. Now we describe the so(7) form of the remaining (anti)commutators given in (2.2). The supercharges with superscript $+$ $(-)$ have positive (negative) $J^{+-}$ charge

$$[J^{+-}, Q^{+R,L}] = \frac{1}{2} Q^{+R,L} \,, \quad [J^{+-}, Q^{-R,L}] = -\frac{1}{2} Q^{-R,L} \tag{2.7}$$

and the superscripts $R$ and $L$ are used to indicate $J^{RL}$ charge:

$$[J^{RL}, Q^{\pm R}] = \frac{1}{2} Q^{\pm R} \,, \qquad [J^{RL}, Q^{\pm L}] = -\frac{1}{2} Q^{\pm L} \,. \tag{2.8}$$

Transformation properties of supercharges with respect to $so(7)$ algebra are given by

$$[J^{ij}, Q^{\pm R,L}] = -\frac{1}{2} \gamma^{ij} Q^{\pm R,L} \,. \tag{2.9}$$

Remaining commutation relations between supercharges and even part of superalgebra take the following form

$$[J^{Li}, Q^{\pm R}] = -\frac{1}{\sqrt{2}} \gamma^i Q^{\pm L} \,, \qquad [J^{Ri}, Q^{\pm L}] = \frac{1}{\sqrt{2}} \gamma^i Q^{\pm R} \,, \tag{2.10}$$

$$[J^{\pm R},\, Q^{\mp L}] = \pm Q^{\pm R} \,, \qquad [J^{\pm L},\, Q^{\mp R}] = \pm Q^{\pm L} \,, \tag{2.11}$$

$$[J^{\pm i}, Q^{\mp R}] = \mp \frac{1}{\sqrt{2}} \gamma^i Q^{\pm R} \,, \qquad [J^{\pm i}, Q^{\mp L}] = \pm \frac{1}{\sqrt{2}} \gamma^i Q^{\pm L} \,. \tag{2.12}$$

In Eqs.(2.10),(2.12) and below $\gamma^i$ stands for $so(7)$ Dirac matrices. Anticommutation relations between supercharges are

$$\{Q^{\pm R},\, Q^{\pm L}\} = \pm P^{\pm}, \qquad \{Q^{+R},\, Q^{-R}\} = P^R, \qquad \{Q^{+L},\, Q^{-L}\} = P^L \,, \tag{2.13}$$

$$\{Q^{\pm L},\, Q^{\mp R}\} = \frac{1}{\sqrt{2}} \gamma^i P^i \,. \tag{2.14}$$

Hermitian conjugation rules in the $so(7)$ basis take the form

$$\begin{gathered} P^{\pm\dagger} = P^{\pm}, \quad P^{i\dagger} = P^i, \quad P^{R\dagger} = P^L, \quad Q^{\pm R\dagger} = Q^{\pm L} \,, \\ J^{ij\dagger} = -J^{ij}, \quad J^{\pm R\dagger} = -J^{\pm L}, \quad J^{RL\dagger} = J^{RL} \,. \end{gathered} \tag{2.15}$$

Next step is to find a realization of Poincaré superalgebra on the space of $11d$ supergravity fields. To do that we use light cone superspace formalism. First, we introduce light cone superspace that is based on position coordinates $x^\mu$ and Grassmann position coordinates $\theta^\alpha$. Second, on this light cone superspace we introduce a scalar superfield $\Phi(x^\mu, \theta)$. In the remainder of this paper we find it convenient to Fourier transform[5] to momentum space for all coordinates except for the time $x^+$. This implies using $p^+$, $p^R$, $p^L$, $p^i$, $\lambda^\alpha$, instead of $x^-$,

[5]Normalization of the Fourier transformation we use is given in formula (E.1).

$x^L$, $x^R$, $x^i$, $\theta^\alpha$ respectively. Thus we consider the scalar superfield $\Phi(x^+, p^+, p^R, p^L, p^i, \lambda)$ with the following expansion in powers of the Grassmann momenta $\lambda$

$$\begin{aligned}\Phi(p,\lambda) &= \beta^2 A + \beta\lambda^\alpha\psi^\alpha + \beta\lambda^{\alpha_1}\lambda^{\alpha_2}A^{\alpha_1\alpha_2} \\ &\quad +\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_3}\psi^{\alpha_1\alpha_2\alpha_3} + \lambda^{\alpha_1}\dots\lambda^{\alpha_4}A^{\alpha_1\dots\alpha_4} + \frac{1}{\beta}(\epsilon\lambda^5)^{\alpha_1\alpha_2\alpha_3}\psi^{\alpha_1\alpha_2\alpha_3 *} \\ &\quad -\frac{1}{\beta}(\epsilon\lambda^6)^{\alpha_1\alpha_2}A^{\alpha_1\alpha_2 *} - \frac{1}{\beta^2}(\epsilon\lambda^7)^\alpha\psi^{\alpha *} + \frac{1}{\beta^2}(\epsilon\lambda^8)A^*\,, \end{aligned} \tag{2.16}$$

where we use the notation[6]

$$\beta \equiv p^+\,, \qquad (\epsilon\lambda^{8-n})^{\alpha_1\dots\alpha_n} \equiv \frac{1}{(8-n)!}\epsilon^{\alpha_1\dots\alpha_n\alpha_{n+1}\dots\alpha_8}\lambda^{\alpha_{n+1}}\dots\lambda^{\alpha_8} \tag{2.17}$$

and $\epsilon^{\alpha_1\dots\alpha_8}$ is the Levi-Civita symbol. In (2.16) the fields and their Hermitian conjugated are related as $(A^*(p))^* = A(-p)$. The superfield $\Phi$ satisfies the reality constraint[7]

$$\Phi(-p,\lambda) = \beta^4 \int d^8\lambda^\dagger e^{\lambda\lambda^\dagger/\beta}(\Phi(p,\lambda))^\dagger\,, \tag{2.18}$$

where for odd variables $F$, (i.e. Grassmann variables and fermionic fields) we use the convention $(F_1F_2)^\dagger = F_2^\dagger F_1^\dagger$. In (2.16) the component fields carrying even number of spinor indices are bosonic fields

$$A^{\alpha_1\dots\alpha_4}(70) = \{h^{ij}(27^0), h^{RL}(1^0), C^{ijk}(35^0), C^{RLi}(7^0)\}\,, \tag{2.19}$$

$$A^{\alpha_1\alpha_2}(28) = \{h^{Li}(7^{-1}), C^{Lij}(21^{-1})\}\,, \tag{2.20}$$

while the fields with odd number of spinor indices are responsible for gravitino field. Explicitly these fields are related as follows[8]

$$\begin{aligned} A_{\alpha_1\dots\alpha_4} &= \frac{1}{2^4\sqrt{2}}\gamma^i_{[\alpha_1\alpha_2}\gamma^j_{\alpha_3\alpha_4]}h^{ij} - \frac{1}{7\cdot 2^3\sqrt{2}}\gamma^i_{[\alpha_1\alpha_2}\gamma^i_{\alpha_4\alpha_4]}h^{RL} \\ &\quad +\frac{1}{3\cdot 2^4\sqrt{2}}\gamma^i_{[\alpha_1\alpha_2}\gamma^{jk}_{\alpha_3\alpha_4]}C^{ijk} + \frac{1}{3\cdot 2^4\sqrt{2}}\gamma^i_{[\alpha_1\alpha_2}\gamma^{ij}_{\alpha_3\alpha_4]}C^{RLj}\,, \end{aligned} \tag{2.21}$$

$$A_{\alpha_1\alpha_2} = -\frac{1}{4}\gamma^i_{\alpha_1\alpha_2}h^{Li} - \frac{1}{8}\gamma^{ij}_{\alpha_1\alpha_2}C^{Lij}\,, \tag{2.22}$$

$$A = \frac{1}{\sqrt{2}}h^{LL}\,, \tag{2.23}$$

[6] In what follows a momentum $p$ as argument of the superfield $\Phi$ and $\delta$- functions designates the set $\{p^I\,, \beta\}$. Also we do not show explicitly the dependence of the superfield on evolution parameter $x^+$. Expansion like (2.16) was introduced for the first time in [5, 6] to study the light cone formulation of $IIB$ supergravity.

[7] Integration measure w.r.t. Grassmann variables is normalized to be $\int d^8\lambda(\epsilon\lambda^8) = 1$. This implies that Grassmann $\delta$-function is given by $\delta(\lambda) = (\epsilon\lambda^8)$.

[8] $[\alpha_1\dots\alpha_n]$ stands for antisymmetrization in $\alpha_1,\dots,\alpha_n$ involving $n!$ terms with overall normalization factor equal to $\frac{1}{n!}$. Graviton field $h^{IJ}$ being $so(9)$ tensor field decomposes into $so(7)$ fields as: $h^{ij}$, $h^{RL}$, $h^{Ri}$, $h^{Li}$. We assume $h^{II} = 0$, $h^{ii} = 0$. Antisymmetric $so(9)$ tensor field $C^{IJK}$ decomposes into $so(7)$ fields in obvious way: $C^{ijk}$, $C^{Rij}$, $C^{Lij}$, $C^{RLi}$. Interrelation of gravitino field components in (2.24),(2.25) with those of the so(9) basis is discussed in Appendix A.

$$\psi_{\alpha_1\alpha_2\alpha_3} = \frac{1}{2\sqrt{2}}\gamma^i_{[\alpha_1\alpha_2}\psi^{\oplus L}_{\alpha_3]i}\,, \tag{2.24}$$

$$\psi_\alpha \quad = \quad \psi^{\oplus L}_{R\alpha}\,. \tag{2.25}$$

Now a representation of the kinematical generators in terms of differential operators acting on the superfield $\Phi$ is given by[9]

$$P^+ = \beta\,, \qquad P^I = p^I\,, \qquad Q^{+R} = \beta\theta\,, \qquad Q^{+L} = \lambda\,, \tag{2.26}$$

$$J^{+I} = \partial_{p^I}\beta\,, \tag{2.27}$$

$$J^{+-} = \partial_\beta\beta - \frac{1}{2}\theta\lambda + 2\,, \tag{2.28}$$

$$J^{ij} = p^i\partial_{p^j} - p^j\partial_{p^i} + \frac{1}{2}\theta\gamma^{ij}\lambda\,, \tag{2.29}$$

$$J^{RL} = p^R\partial_{p^R} - p^L\partial_{p^L} + \frac{1}{2}\theta\lambda - 2\,, \tag{2.30}$$

$$J^{Ri} = p^R\partial_{p^i} - p^i\partial_{p^L} - \frac{1}{2\sqrt{2}}\beta\theta\gamma^i\theta\,, \tag{2.31}$$

$$J^{Li} = p^L\partial_{p^i} - p^i\partial_{p^R} + \frac{1}{2\sqrt{2}\beta}\lambda\gamma^i\lambda\,. \tag{2.32}$$

Here and below we use the notation

$$\partial_\beta \equiv \partial/\partial\beta\,, \quad \partial_{p^i} \equiv \partial/\partial p^i\,, \quad \partial_{p^R} \equiv \partial/\partial p^R\,, \quad \partial_{p^L} \equiv \partial/\partial p^L\,, \qquad \not{p} \equiv \gamma^i p^i\,,$$
$$\gamma^{ij} \equiv \frac{1}{2}\gamma^i\gamma^j - (i \leftrightarrow j)\,. \tag{2.33}$$

Representation of the dynamical generators in terms of differential operators acting on the superfield $\Phi$ is given by

$$P^- = p^-\,, \qquad p^- \equiv -\frac{p^Ip^I}{2\beta}\,, \tag{2.34}$$

$$J^{-R} = \partial_{p^L}p^- - \partial_\beta p^R - \frac{1}{2\sqrt{2}}\theta\not{p}\theta + \frac{1}{\beta}p^R\theta\lambda - \frac{4}{\beta}p^R\,, \tag{2.35}$$

$$J^{-L} = \partial_{p^R}p^- - \partial_\beta p^L + \frac{1}{2\sqrt{2}\beta^2}\lambda\not{p}\lambda\,, \tag{2.36}$$

$$J^{-i} = \partial_{p^i}p^- - \partial_\beta p^i + \frac{1}{2\beta}\theta\gamma^i\not{p}\lambda - \frac{1}{2\sqrt{2}\beta^2}p^R\lambda\gamma^i\lambda + \frac{1}{2\sqrt{2}}p^L\theta\gamma^i\theta - \frac{2}{\beta}p^i\,, \tag{2.37}$$

$$Q^{-R} = \frac{1}{\sqrt{2}}\theta\not{p} + \frac{1}{\beta}p^R\lambda\,, \tag{2.38}$$

$$Q^{-L} = p^L\theta + \frac{1}{\sqrt{2}\beta}\not{p}\lambda\,. \tag{2.39}$$

[9]Throughout this paper without loss of generality we analyze generators of Poincaré superalgebra and their commutators for $x^+ = 0$.

The Grassmann coordinates $\theta$ and momenta $\lambda$ satisfy the following anticommutation and hermitian conjugations rules

$$\{\lambda^{\alpha_1},\theta^{\alpha_2}\} = \delta^{\alpha_1\alpha_2}\,, \qquad \lambda^\dagger = p^+\theta\,, \qquad \theta^\dagger = \frac{1}{p^+}\lambda\,. \tag{2.40}$$

The above-given expressions provide realization of Poincaré superalgebra in terms of differential operators acting on the physical superfield $\Phi$. Now let us write down a field theoretical realization of this algebra in terms of the physical superfield $\Phi$. As we mentioned above the kinematical generators $\hat{G}^{kin}$ are realized quadratically in $\Phi$, while the dynamical generators $\hat{G}^{dyn}$ are realized non-linearly. At a quadratical level both $\hat{G}^{kin}$ and $\hat{G}^{dyn}$ admit the following representation

$$\hat{G} = \int \beta d^{10}p\, d^8\lambda \Phi(-p,-\lambda) G\Phi(p,\lambda)\,, \qquad d^{10}p \equiv d\beta d^9 p\,, \tag{2.41}$$

where $G$ are the differential operators given above in (2.26)-(2.39). The field $\Phi$ satisfies the Poisson-Dirac commutation relation

$$[\Phi(p,\lambda),\, \Phi(p',\lambda')]\Big|_{equal\, x^+} = \frac{\delta^{10}(p+p')}{2\beta}\delta^8(\lambda+\lambda')\,. \tag{2.42}$$

With these definitions one has the standard commutation relation

$$[\Phi,\hat{G}\,] = G\Phi\,. \tag{2.43}$$

Note that our normalization of the component fields in expansion of the superfield $\Phi$ (see (2.16) and (2.21)-(2.25)) is chosen so that contributions of component fields to the generators, say for $P^+$, are weighted as follows

$$P^+ = \int d^{10}p\ \beta^2 (\frac{1}{2}h^{IJ}(-p)h^{IJ}(p) + \frac{1}{3!}C^{IJK}(-p)C^{IJK}(p) + \psi_I^\oplus(-p)C_{16}\psi_I^\oplus(p))\,, \tag{2.44}$$

where we used a notation of the $so(9)$ basis. Light-cone gauge action takes then the following standard form

$$S = \int dx^+ d^{10}p\, d^8\lambda\ \Phi(-p,-\lambda)\mathrm{i}\,\beta\partial^-\Phi(p,\lambda) + \int dx^+ P^-\,. \tag{2.45}$$

This representation for the light cone action is valid both for free and interacting theory. Hamiltonian of free theory can be obtained from Eqs.(2.34),(2.41).

# 3 General structure of $n$-point interaction vertices

We begin with discussion of the general structure of Poincaré superalgebra dynamical generators (2.6). In theories of interacting fields the dynamical generators receive corrections having higher powers of physical fields and one has the following expansion for them

$$G^{dyn} = \sum_{n=2}^\infty G^{dyn}_{(n)}\,, \tag{3.1}$$

where $G^{dyn}_{(n)}$ stands for $n$ - point contribution (degree $n$ in physical fields) to the dynamical generators. The generators $G^{dyn}$ of classical SYM theories do not receive corrections higher than fourth order in fields [16, 17, 18], while the generators $G^{dyn}_{(n)}$ for supergravity theories are nontrivial for all $n \geq 2$ [19, 20, 21][10].

The 'free' generators $G^{dyn}_{(2)}$ (3.1), which are quadratical in fields, were discussed in the preceding section (see (2.41)). In this section we discuss general structure of 'interacting' dynamical generators $G^{dyn}_{(n)}$, $n \geq 3$. Namely, we describe those properties of the dynamical generators $G^{dyn}_{(n)}$, $n \geq 3$ that can be obtained from commutation relations between $G^{kin}$ and $G^{dyn}$. In other words we find restrictions imposed by kinematical symmetries on the dynamical 'interacting' generators. We proceed in the following way.

(**i**) First of all we consider restrictions imposed by kinematical symmetries on the following dynamical generators

$$P^-\,, \quad Q^{-R}\,, \quad Q^{-L}\,. \tag{3.2}$$

As seen from (anti)commutators (2.1),(2.7)-(2.14) all kinematical generators (2.5) with exception of $J^{+-}$, $J^{IJ}$ have the following commutation relations with dynamical generators (3.2): $[G^{dyn}, G^{kin}] = G^{kin}$. Because $G^{kin}$ are quadratic in fields we get from this the (anti)commutation relations

$$[G^{dyn}_{(n)}, G^{kin}] = 0\,, \qquad n \geq 3\,. \tag{3.3}$$

Exploiting (3.3) for $G^{kin} = (P^I, P^+, Q^{+L})$ we get the following representation for the dynamical generators (3.2):

$$P^-_{(n)} = \int d\Gamma_n \Phi_{(n)} p^-_{(n)}\,, \tag{3.4}$$

$$Q^{-R,L}_{(n)} = \int d\Gamma_n \Phi_{(n)} q^{-R,L}_{(n)}\,, \tag{3.5}$$

where we use the notation

$$\Phi_{(n)} \equiv \prod_{a=1}^n \Phi(p_a, \lambda_a)\,, \tag{3.6}$$

$$d\Gamma_n \equiv d\Gamma_n(p) d\Gamma_n(\lambda)\,, \tag{3.7}$$

$$d\Gamma_n(p) \equiv (2\pi)^{d-1}\delta^{d-1}(\sum_{a=1}^n p_a)\prod_{a=1}^n \frac{d^{d-1}p_a}{(2\pi)^{(d-1)/2}}\,, \qquad d = 11\,, \tag{3.8}$$

$$d\Gamma_n(\lambda) \equiv \delta^8(\sum_{a=1}^n \lambda_a)\prod_{a=1}^n d^8\lambda_a\,. \tag{3.9}$$

[10]Generators of closed string field theories, which involve graviton field, terminates at cubic correction $G^{dyn}_3$ [6, 8]. On the other hand it is natural to expect that generators of general covariant theory should involve all powers of graviton field $h_{\mu\nu}$. The fact that closed string field theories do not involve higher than third order vertices in $h_{\mu\nu}$ implies that the general covariance in closed string field theories is realized in a highly nontrivial way. In string theory the general covariance manifests itself upon integration out of massive string modes and going to the low energy expansion. See [22] for interesting discussion of this theme.

Densities $p^-_{(n)}$, $q^{-R,L}_{(n)}$ in (3.4),(3.5) depend on light cone momenta $\beta_a$, transverse momenta $p^I_a$ and Grassmann momenta $\lambda_a$. The $\delta$- functions in (3.8),(3.9) respect conservation laws for these momenta. Here and below the indices $a,b=1,\ldots n$ label $n$ interacting fields.

(**ii**) Making use of (3.3) for $G^{kin}=(J^{+I},Q^{+R})$ we find that the densities $p^-_{(n)}$, $q^{-R,L}_{(n)}$ depend on momenta $p^I_a$ and $\lambda_a$ through the following quantities

$$\mathbb{P}^I_{ab}\equiv p^I_a\beta_b-p^I_b\beta_a\,,\qquad \Lambda_{ab}\equiv\lambda_a\beta_b-\lambda_b\beta_a\,, \tag{3.10}$$

*i.e.* the densities $p^-_{(n)}$, $q^{-R,L}_{(n)}$ turn out to be functions of $\mathbb{P}^I_{ab}$, $\Lambda_{ab}$[11] instead of $p^I_a$, $\lambda_a$:

$$p^-_{(n)}(p_a,\lambda_a,\beta_a)=p^-_{(n)}(\mathbb{P}_{ab},\Lambda_{ab},\beta_a)\,, \tag{3.11}$$

$$q^{-R,L}_{(n)}(p_a,\lambda_a,\beta_a)=q^{-R,L}_{(n)}(\mathbb{P}_{ab},\Lambda_{ab},\beta_a)\,. \tag{3.12}$$

(**iii**) Commutators between $G^{dyn}$ and remaining kinematical generators $G^{kin}=(J^{+-},J^{IJ})$ have the form $[G^{kin},G^{dyn}]=G^{dyn}$. Because $G^{kin}$ are quadratic in physical fields, *i.e.* $G^{kin}_{(n)}=0$ for $n>2$, we get the important commutation relations between above mentioned $G^{kin}$ and all $G^{dyn}$ to be written schematically as

$$[G^{dyn}_{(n)},G^{kin}]=G^{dyn}_{(n)}\,,\qquad n\geq 2\,. \tag{3.13}$$

Before to proceed we introduce $j^{+-}$- and $j^{RL}$- charges by relations[12]

$$[J^{+-},G]=j^{+-}G\,,\qquad [J^{RL},G]=j^{RL}G\,. \tag{3.14}$$

It is straightforward to check that commutators (3.13) taken for $G^{kin}=J^{+-},J^{RL}$ lead to the following respective equations for densities $g_{(n)}=p^-_{(n)}$, $q^{-R,L}_{(n)}$:

$$\sum_{a=1}^n(\beta_a\partial_{\beta_a}+\frac{1}{2}\lambda_a\partial_{\lambda_a})g_{(n)}=(2n+j^{+-}-3)g_{(n)}\,, \tag{3.15}$$

$$\sum_{a=1}^n(p^L_a\partial_{p^L_a}-p^R_a\partial_{p^R_a}+\frac{1}{2}\lambda_a\partial_{\lambda_a})g_{(n)}=(2n-j^{RL}-4)g_{(n)}\,. \tag{3.16}$$

Remaining equations we will need below can be obtained from commutators (3.13) taken for $G^{dyn}=P^-$ and $G^{kin}=J^{Ri},J^{Li}$. For these generators the commutators (3.13) take a form $[P^-,G^{kin}]=0$ and the latter commutators lead to the following equations for density $p^-_{(n)}$:

$$\sum_{a=1}^n(p^i_a\partial_{p^L_a}-p^R_a\partial_{p^i_a}-\frac{1}{2\sqrt{2}}\beta_a\theta_a\gamma^i\theta_a)p^-_{(n)}=0\,, \tag{3.17}$$

$$\sum_{a=1}^n(p^i_a\partial_{p^R_a}-p^L_a\partial_{p^i_a}+\frac{1}{2\sqrt{2}\beta_a}\lambda_a\gamma^i\lambda_a)p^-_{(n)}=0\,. \tag{3.18}$$

[11] Note that due to momentum conservation laws not all $\mathbb{P}^I_{ab}$, (and $\Lambda_{ab}$) are independent. It easy to check that $n$-point vertex involves $n-2$ independent 'momenta' $\mathbb{P}^I_{ab}$ (and $\Lambda_{ab}$).

[12] $j^{+-}$- and $j^{RL}$- charges can be easily obtained from commutators (2.1),(2.7) and (2.8).

(**iv**) To complete a description of the dynamical generators we should consider the dynamical generator $J^{-I}$. Making use of commutation relations of $J^{-I}$ with the kinematical generators we get the following representation for $J^{-I}$:

$$J^{-R}_{(n)} = \int d\Gamma_n \Big( \Phi_{(n)} j^{-R}_{(n)} + \frac{1}{n} (\sum_{a=1}^n \partial_{p^L_a} \Phi_{(n)}) p^-_{(n)} + \frac{1}{n} (\sum_{a=1}^n \theta_a \Phi_{(n)}) q^{-R}_{(n)} \Big) , \tag{3.19}$$

$$J^{-L}_{(n)} = \int d\Gamma_n \Big( \Phi_{(n)} j^{-L}_{(n)} + \frac{1}{n} (\sum_{a=1}^n \partial_{p^R_a} \Phi_{(n)}) p^-_{(n)} + \frac{1}{n} \Phi_{(n)} \sum_{a=1}^n \frac{\lambda_a}{\beta_a} q^{-L}_{(n)} \Big) , \tag{3.20}$$

$$\begin{aligned} J^{-i}_{(n)} = & \int d\Gamma_n \Big( \Phi_{(n)} j^{-i}_{(n)} + \frac{1}{n} (\sum_{a=1}^n \partial_{p^i_a} \Phi_{(n)}) p^-_{(n)} \\ & + \frac{1}{\sqrt{2} n} (\sum_{a=1}^n \theta_a \gamma^i \Phi_{(n)}) q^{-L}_{(n)} - \frac{1}{\sqrt{2} n} \Phi_{(n)} \sum_{a=1}^n \frac{\lambda_a}{\beta_a} \gamma^i q^{-R}_{(n)} \Big) . \end{aligned} \tag{3.21}$$

Here we introduce new densities $j^{-I}_{(n)}$. Commutation relations of $J^{-I}$ with the kinematical generators tell us that $j^{-I}_{(n)}$ depend on $\beta_a$ and momenta $\mathbb{P}^I_{ab}$, $\Lambda_{ab}$ (3.10).

To summarize, commutation relations between the kinematical and dynamical generators give us the expressions for dynamical generators (3.4),(3.5),(3.19)-(3.21), where all densities $p^-_{(n)}$, $q^{-R,L}_{(n)}$, $j^{-I}_{(n)}$ depend on $\mathbb{P}^I_{ab}$, $\Lambda_{ab}$ and the densities satisfy the equations (3.15)-(3.18).

In order to fix the densities $p^-_{(n)}$, $q^{-R,L}_{(n)}$, $j^{-I}_{(n)}$ we should consider commutation relations between respective dynamical generators and a general strategy of finding these densities consists basically of the following two steps:

- First step is to find restrictions imposed by commutation relations of Poincaré superalgebra between dynamical generators. Usually from these commutation relations one learns that not all densities are independent. It turns out that density $p^-_{(n)}$ is still to be independent, while all remaining densities, *i.e.* $q^{-R,L}_{(n)}$, $j^{-I}_{(n)}$, are expressible in terms of $p^-_{(n)}$.

- Second step is to find solution to the independent density $p^-_{(n)}$. The solution is found, as we will demonstrate below, from the requirement that all densities, *i.e.* $p^-_{(n)}$, $q^{-R,L}_{(n)}$, $j^{-I}_{(n)}$, be polynomial in the transverse momenta $p^I_a$ (and Grassmann momenta $\lambda_a$ as well). This requirement we shall refer to as a locality condition.

Below we apply this strategy to study 3- and 4-point interaction vertices.

# 4 Cubic interaction vertices

Although many examples of cubic vertices are known in the literature, constructing cubic vertices for concrete field theoretical models is still a challenging procedure. The general method essentially simplifying the procedure of obtaining cubic interaction vertices was discovered in [11], developed in [12, 13] and formulated finally in [10]. One of the characteristic features of this method is reducing manifest transverse $so(d-2)$ invariance (which is $so(9)$

for $11d$ supergravity) to $so(d-4)$ invariance (which is $so(7)$ in this paper)[13]. On the other hand, it is the $so(7)$ symmetries that are manifest symmetries of the unconstrained superfield formulation of $11d$ supergravity. In other words the manifest symmetries of our method and those of the unconstrained superfield formulation of $11d$ supergravity match. In this section we would like to demonstrate how the method of Ref.[10] works for the case of $11d$ supergravity.

As was explained above (see (3.11)) cubic vertex $p_{(3)}^-$ depends on the 'momenta' $\mathbb{P}_{ab}$, $\Lambda_{ab}$ and $\beta_a$, where $a,b=1,2,3$ label three interacting fields in cubic vertex. The variables $\mathbb{P}_{12}^I$, $\mathbb{P}_{23}^I$, $\mathbb{P}_{31}^I$ however are not independent: all of them are expressible in terms of $\mathbb{P}^I$ defined by[14]

$$\mathbb{P}^I = \frac{1}{3}\sum_{a=1}^{3}\check{\beta}_a p_a^I\,, \qquad \check{\beta}_a \equiv \beta_{a+1}-\beta_{a+2}\,, \quad \beta_a\equiv\beta_{a+3}\,. \tag{4.1}$$

The same holds for Grassmann variables $\Lambda_{ab}$, *i.e.* due to momentum conservation laws for $\beta_a$ and Grassmann momenta $\lambda_a$ the variables $\Lambda_{12}$, $\Lambda_{23}$ $\Lambda_{31}$ (see 3.10) are also expressible in terms of the variable $\Lambda$ defined by

$$\Lambda = \frac{1}{3}\sum_{a=1}^{3}\check{\beta}_a\lambda_a\,. \tag{4.2}$$

The usage of $\mathbb{P}^I$ and $\Lambda$ is advantageous since they are manifestly invariant under cyclic permutations of indices $1,2,3$, which label three interacting fields. Thus the vertex $p_{(3)}^-$ is eventually function of $\mathbb{P}^I$, $\Lambda$ and $\beta_a$:

$$p_{(3)}^- = p_{(3)}^-(\mathbb{P},\Lambda,\beta_a)\,. \tag{4.3}$$

In general the vertex $p_{(3)}^-$ is a monomial of degree $k$ in $\mathbb{P}^I$. As is well know the generic $11d$ supergravity is described by vertex $p_{(3)}^-$, which is monomial of degree two in transverse 'momentum' $\mathbb{P}^I$, i.e. have to set $k=2$. However for flexibility we keep $k$ to be arbitrary. By doing this we will be able to demonstrate explicitly the fact that cubic vertices corresponding to $k=4,6$, which are responsible for the higher derivative $R^2$ and $R^3$- terms ($R$ stands for Riemann tensor), do not admit supersymmetric extension.

The method of finding cubic vertices suggested in [10] consists of the following steps:

(**i**) First we find dependence of the vertex $p_{(3)}^-$ on 'momentum' $\mathbb{P}^I$. To this end we use commutation relations $[P^-,J^{IJ}]=0$, which lead to the following equations for $p_{(3)}^-$:

$$J^{IJ}(\mathbb{P},\Lambda)p_{(3)}^- = 0\,, \qquad J^{IJ}(\mathbb{P},\Lambda)\equiv L^{IJ}(\mathbb{P})+M^{IJ}(\Lambda)\,, \tag{4.4}$$

where orbital part of angular momentum is given by

$$L^{IJ}(\mathbb{P}) \equiv \mathbb{P}^I\partial_{\mathbb{P}^J} - \mathbb{P}^J\partial_{\mathbb{P}^I}\,, \tag{4.5}$$

[13]In the preceding studies [8], reducing the manifest $so(d-2)$ symmetry to $so(d-4)$ was used to formulate superfield theory of $IIA$ superstrings. In the latter reference the reducing was motivated by desire to get the unconstrained superfield formulation. In [10] the main motivation for reducing was desire to get the most general solution for cubic vertex for arbitrary spin fields of (super) Poincaré invariant theory. Discussion of the $so(7)$ formalism in the context of M(atrix) theory can be found in [23].

[14]By using momentum conservation laws for $p_a^I$ and $\beta_a$ it is easy to check that $\mathbb{P}_{12}^I=\mathbb{P}_{23}^I=\mathbb{P}_{31}^I=\mathbb{P}^I$.

while spin operators are given by

$$M^{RL}(\Lambda) = \frac{1}{2}\theta_\Lambda\Lambda - 2\,, \tag{4.6}$$

$$M^{ij}(\Lambda) = \frac{1}{2}\theta_\Lambda\gamma^{ij}\Lambda\,, \tag{4.7}$$

$$M^{Ri}(\Lambda) = -\frac{1}{2\sqrt{2}}\hat\beta\theta_\Lambda\gamma^i\theta_\Lambda\,, \tag{4.8}$$

$$M^{Li}(\Lambda) = \frac{1}{2\sqrt{2}\hat\beta}\Lambda\gamma^i\Lambda\,. \tag{4.9}$$

Here and below we use the notation

$$\hat\beta \equiv \beta_1\beta_2\beta_3\,, \tag{4.10}$$

and $\theta_\Lambda$ is a derivative with respect to $\Lambda$:

$$\theta_\Lambda \equiv \partial_\Lambda\,, \qquad \{\theta_\Lambda, \Lambda\} = 1\,. \tag{4.11}$$

In what follows we prefer to exploit instead of 'momenta' $\mathbb{P}^I = (\mathbb{P}^L, \mathbb{P}^R, \mathbb{P}^i)$ a dimensionfull 'momentum' $\mathbb{P}^L$ and dimensionless variables $q^i$, $\rho$ defined by

$$q^i \equiv \frac{\mathbb{P}^i}{\mathbb{P}^L}\,, \qquad \rho \equiv \frac{\mathbb{P}^i\mathbb{P}^i + 2\mathbb{P}^R\mathbb{P}^L}{2(\mathbb{P}^L)^2}\,, \qquad \frac{\mathbb{P}^R}{\mathbb{P}^L} = \rho - \frac{q^2}{2}\,. \tag{4.12}$$

In terms of the new variables the cubic interaction vertex can be cast into the form

$$p_{(3)}^- = (\mathbb{P}^L)^k V(q\,, \rho\,, \beta\,, \Lambda)\,, \tag{4.13}$$

which demonstrates explicitly that the vertex $p_{(3)}^-$ is a monomial of degree $k$ in transverse 'momentum' $\mathbb{P}^I$. In terms of new variables various components of the orbital momentum operator (4.5) take the following form

$$L^{RL} = q\partial_q + 2\rho\partial_\rho - \mathbb{P}^L\partial_{\mathbb{P}^L}\,, \tag{4.14}$$

$$L^{ij} = q^i\partial_{q^j} - q^j\partial_{q^i}\,, \tag{4.15}$$

$$L^{Li} = \partial_{q^i}\,, \tag{4.16}$$

$$L^{Ri} = (\rho - \frac{q^2}{2})\partial_{q^i} + q^i(q\partial_q + 2\rho\partial_\rho - \mathbb{P}^L\partial_{\mathbb{P}^L})\,. \tag{4.17}$$

To demonstrate main idea of introducing the variable $q^i$ let us focus on the $Li$ part of Eqs.(4.4). Plugging in the $Li$ part of Eqs.(4.4) the respective representations for $p_{(3)}^-$ and $L^{Li}$ in (4.13) and (4.16) we get

$$(\partial_{q^i} + M^{Li}(\Lambda))V = 0\,, \tag{4.18}$$

where $M^{Li}$ is given in (4.9). Solution of (4.18) is easily found to be

$$V(q\,, \rho\,, \beta\,, \Lambda) = E_q\tilde V(\rho\,, \beta\,, \Lambda)\,, \tag{4.19}$$

where an operator $E_q$ is defined to be

$$E_q \equiv \exp(-q^j M^{Lj}(\Lambda))\,. \tag{4.20}$$

Thus collecting above-given expressions we get the following intermediate representation for the vertex $p_{(3)}^-$:

$$p_{(3)}^- = (\mathbb{P}^L)^k E_q \tilde{V}(\rho\,,\beta\,,\Lambda)\,. \tag{4.21}$$

Next step is to find dependence on variable $\rho$. To do that we use $LR$, $Ri$ and $ij$ parts of Eqs.(4.4). Details of derivation may be found in Appendix B and our result is given by

$$\tilde{V}(\rho,\beta,\Lambda) = E_\rho \tilde{V}_0(\beta,\Lambda)\,, \tag{4.22}$$

where an operator $E_\rho$ is defined by relation

$$E_\rho \equiv \sum_{n=0}^{k}(-\rho)^n \frac{\Gamma(\frac{7}{2}+k-n)}{2^n n!\Gamma(\frac{7}{2}+k)}(M^{Lj}(\Lambda)M^{Lj}(\Lambda))^n \tag{4.23}$$

and new vertex $\tilde{V}_0$ satisfies the following equations

$$(M^{RL}(\Lambda)-k)\tilde{V}_0 = 0\,, \tag{4.24}$$

$$M^{Ri}(\Lambda)\tilde{V}_0 = 0\,, \tag{4.25}$$

$$M^{ij}(\Lambda)\tilde{V}_0 = 0\,. \tag{4.26}$$

As seen from (4.22) the vertex $\tilde{V}_0$ depends only on Grassmann 'momentum' $\Lambda$ and light cone momenta $\beta_a$. The dependence on the transverse space 'momentum' $\mathbb{P}^I$ is thus fixed explicitly and we get the following representation for the cubic vertex

$$p_{(3)}^-(\mathbb{P},\Lambda,\beta_a) = (\mathbb{P}^L)^k E_q E_\rho \tilde{V}_0(\Lambda,\beta_a)\,, \tag{4.27}$$

where $\tilde{V}_0$ satisfies Eqs.(4.24)-(4.26). We note that while deriving this representation we used general form of the orbital momentum $L^{IJ}$ (4.4), which is valid for arbitrary Poincaré invariant theory. Therefore the representation for the vertex $p_{(3)}^-$ given in (4.27) is universal and is valid for arbitrary Poincaré invariant theory. Various theories differ by: (i) spin operators in angular momentum (for the case under consideration these spin operators are given in (4.6)-(4.9)); (ii) the vertex $\tilde{V}_0$, which for case under consideration depends on Grassmann 'momentum' $\Lambda$ and light cone momenta $\beta_a$. Now we proceed to the second step of our method.

(**ii**) At this stage we find dependence on Grassmann 'momentum' $\Lambda$. To this end we use Eqs.(4.24)-(4.25), which turn out to be very simple to analyze. Indeed, making use of expression for $M^{RL}(\Lambda)$ given in (4.6) we find the equation

$$\Lambda\theta_\Lambda \tilde{V}_0 = 2(2-k)\tilde{V}_0\,. \tag{4.28}$$

An operator $\Lambda\theta_\Lambda$ counts power of Grassmann 'momentum' $\Lambda$ involved in the vertex $\tilde{V}_0$ that cannot involve terms having negative power of $\Lambda$, *i.e.* eigenvalues of the operator $\Lambda\theta_\Lambda$ must be non-negative. Eq.(4.28) implies then that vertices with terms higher than second order

in $\mathbb{P}^I$, i.e. with $k > 2$, are forbidden. Note that the values $k = 4$ and $k = 6$ correspond to the $R^2$- and $R^3$- terms. Therefore, the fact that vertices with $k = 4$ and $k = 6$ are forbidden implies that the $R^2$ and $R^3$- terms do not allow supersymmetric extension. One important thing to note is that we proved absence of supersymmetric extension of the $R^2$ and $R^3$- terms by using only commutation relations between Hamiltonian $P^-$ and the kinematical generators. It is reasonable to think that the kinematical generators do not receive quantum corrections. If this indeed would be the case then our result could be considered as light cone proof of non-renormalization of $R^2$ and $R^3$ terms in $11d$ supergravity [15].

The remaining case of $k = 2$ corresponds to cubic vertex of generic $11d$ supergravity[16] and Eq.(4.28) tells us that for $11d$ supergravity the $\tilde{V}_0$ does not depends on $\Lambda$ at all, *i.e.* entire dependence of the cubic vertex $p^-_{(3)}$ (4.27) on 'momenta' $\mathbb{P}^i$ and $\Lambda$ is governed by $E$-operators $E_q$, $E_\rho$ (4.20),(4.23), which are purely algebraic in nature.

(**iiii**) Last step is to find dependence of $\tilde{V}_0$ on three momenta $\beta_a$. Because of conservation law $\sum_{a=1}^3 \beta_a = 0$ the vertex $\tilde{V}_0$ depends on two light cone momenta. Therefore we need two equations to fix $\tilde{V}_0$. One of equations is obtainable from commutator $[P^-, J^{+-}] = P^-$ and was given in (3.15), where we have to set $j^{+-} = -1$, $n = 3$, $g_{(3)} = p^-_{(3)}$. Exploiting then the representation (4.3) we obtain the following equation

$$(\sum_{a=1}^3 \beta_a \partial_{\beta_a} + \frac{3}{2}\Lambda\theta_\Lambda + k - 2) p^-_{(3)} = 0\,, \tag{4.29}$$

which in terms of $\tilde{V}_0$ takes the following form

$$\sum_{a=1}^3 \beta_a \partial_{\beta_a} \tilde{V}_0 = 0\,. \tag{4.30}$$

The second equation for $\tilde{V}_0$ can be found from commutation relations between dynamical generators and requirement we call a locality condition to be formulated precisely below. Namely, making use of commutator $[P^-, J^{-I}] = 0$ and expression for $J^{-I}$ given in (3.19)-(3.21) one can find the following relation (for details see Appendix C)

$$j^{-I}_{(3)} = -\frac{2\mathbb{P}^I}{3|\mathbb{P}|^2} \sum_{a=1}^3 \check{\beta}_a \beta_a \partial_{\beta_a} p^-_{(3)}\,. \tag{4.31}$$

This expression tells us that the density $j^{-I}_{(3)}$ is not independent quantity but expressible in terms of the interacting vertex $p^-_{(3)}$. Remaining commutators between the dynamical generators also do not fix the vertex $p^-_{(3)}$ uniquely. This implies that restrictions imposed by commutation relations of Poincaré superalgebra by themselves are not sufficient to fix the interaction vertex $p^-_{(3)}$ uniquely. To choose physical relevant vertices $p^-_{(3)}$ and $j^{-I}_{(3)}$, i.e. to fix them uniquely, we impose the requirement we refer to as the locality condition: demand the vertices $p^-_{(3)}$, $j^{-I}_{(3)}$ be monomial in $\mathbb{P}^I$. As to the vertex $p^-_{(3)}$ we demand this vertex be local (*i.e.* monomial in $\mathbb{P}^I$) from the very beginning. However from Eq.(4.31) it is clear that local $p^-_{(3)}$ does not lead automatically to local density $j^{-I}_{(3)}$. From the expressions for $\rho$ (see 4.12) and

[15] Discussion of the $R^2$ and $R^3$- terms in string theory effective action may be found in [24].

[16] Note that Eq.(4.28) does not formally rule out the cases of $k = 0, 1$. It is easy to demonstrate however that these cases are ruled out by Eqs.(4.25),(4.26).

formulas (4.23),(4.27) on can demonstrate that the locality condition amounts to requiring the $\tilde{V}_0$ satisfies the equation

$$\sum_{a=1}^{3} \check{\beta}_a \beta_a \partial_{\beta_a} \tilde{V}_0 = 0\,. \tag{4.32}$$

This equation reflects simply the fact that in order to cancel denominator $|\mathbb{P}|^2$ in r.h.s of Eq.(4.31) we have to cancel the contribution of $n = 0$ term to the expansion (4.23). Equations (4.30) and (4.32) tell us that $\tilde{V}_0$ does not depend on momenta $\beta_a$ at all and therefore $\tilde{V}_0$ is fixed to be

$$\tilde{V}_0 = \frac{\kappa}{3}\,, \tag{4.33}$$

where $\kappa$ is gravitational constant (see formula (4.39) below). Thus our final result for the cubic vertex is

$$p_{(3)}^-(\mathbb{P}, \Lambda, \beta) = \frac{\kappa}{3}\, \mathbb{P}^{L\,2} E_q E_\rho\,, \tag{4.34}$$

where the $E$- operators $E_q$ and $E_\rho$ are given by (4.20) and (4.23). Note that in expression for $E_\rho$ corresponding to $11d$ supergravity vertex we have to set $k = 2$ and this gives the expansion

$$E_\rho = 1 - \frac{\rho}{9} M^{Li}(\Lambda) M^{Li}(\Lambda) + \frac{\rho^2}{7 \cdot 18} (M^{Li}(\Lambda) M^{Li}(\Lambda))^2\,. \tag{4.35}$$

Making use of these expressions and formula for $M^{Li}(\Lambda)$ given in (4.9) we can work out an explicit representation for the cubic vertex in a rather straightforward way

$$\begin{aligned} \frac{3}{\kappa} p_{(3)}^- &= \mathbb{P}^{L2} - \frac{\mathbb{P}^L}{2\sqrt{2}\hat{\beta}} \Lambda \not{\mathbb{P}} \Lambda + \frac{1}{16\hat{\beta}^2} (\Lambda \not{\mathbb{P}} \Lambda)^2 - \frac{|\mathbb{P}|^2}{9 \cdot 16 \hat{\beta}^2} (\Lambda \gamma^j \Lambda)^2 \\ &+ \frac{\mathbb{P}^R}{9 \cdot 16\sqrt{2}\hat{\beta}^3} \Lambda \not{\mathbb{P}} \Lambda (\Lambda \gamma^j \Lambda)^2 + \frac{\mathbb{P}^{R2}}{2^7 \cdot 63 \hat{\beta}^4} ((\Lambda \gamma^i \Lambda)^2)^2\,, \end{aligned} \tag{4.36}$$

where throughout this paper we use the notation

$$\not{\mathbb{P}} \equiv \mathbb{P}^i \gamma^i\,, \qquad |\mathbb{P}|^2 \equiv \mathbb{P}^I \mathbb{P}^I\,. \tag{4.37}$$

Thus the action in cubic approximation is given by expression (2.45), where we have to insert Hamiltonian $P^-$ given by (3.4) and (4.34). Note also that formulas (4.31) and (4.36) imply the relation

$$j_{(3)}^{-I} = 0\,. \tag{4.38}$$

We choose normalization (4.33) so that the cubic vertex for graviton field obtainable from our action (2.45) coincides with that of Einstein-Hilbert action taken in the normalization

$$S_{EH} = \int d^{11}x\, \mathcal{L}_{EH}\,, \qquad \mathcal{L}_{EH} \equiv \frac{1}{2\kappa^2} \sqrt{-g}\, R\,. \tag{4.39}$$

Let us make comment how the normalization of our vertex can be related with that of the action (4.39) in a most simple way. Making use an expansion for metric tensor[17]

$$g_{\mu\nu} = \delta_{\mu\nu} + \sqrt{2}\,\kappa h_{\mu\nu} \tag{4.40}$$

[17] We adopt the conventions $R^\mu{}_{\nu\lambda\rho} \equiv \partial_\lambda \Gamma^\mu_{\nu\rho} + \ldots$, $R_{\nu\rho} \equiv R^\mu{}_{\nu\mu\rho}$.

we get the following expansion (up to total derivatives) for Einstein-Hilbert Lagrangian (4.39) in cubic approximation

$$\frac{1}{\sqrt{2}\kappa}\mathcal{L}_{EH}\,|_{(3)} = \frac{1}{4}h_{\mu\nu}\partial_\mu h_{\rho\lambda}\partial_\nu h_{\rho\lambda} + \frac{1}{2}h_{\mu\nu}\partial_\lambda h_{\mu\rho}\partial_\rho h_{\nu\lambda} + \frac{1}{2}h_{\mu\nu}\partial_\rho h_{\nu\lambda}\partial_\rho h_{\lambda\mu}\,. \tag{4.41}$$

To derive this formula we use the constraints $h^\mu_\mu = 0$ and $\partial_\mu h^{\mu\nu} = 0$ as we are going to treat cubic vertices in light cone gauge

$$h^{++} = 0\,, \qquad h^{+-} = 0\,, \qquad h^{+I} = 0\,, \tag{4.42}$$

which leads to such constraints. Making use of light cone gauge (4.42) gives the following solution to non-physical degrees of freedom ($h^{II} = 0$):

$$h^{-I}(p) = -\frac{p^J}{\beta}\,h^{IJ}(p)\,, \qquad h^{--}(p) = \frac{p^I p^J}{\beta^2}\,h^{IJ}(p)\,. \tag{4.43}$$

Plugging (4.42),(4.43) into (4.39) and keeping in mind (2.45) we get Hamiltonian

$$P^-_{(3)} = \sqrt{2}\kappa\int d\Gamma_3(p)\Big(h_1^{KN}h_2^{KN}h_3^{IJ}\frac{\mathbb{P}^I\mathbb{P}^J}{4\beta_3^2} + h_1^{IK}h_2^{JN}h_3^{KN}\frac{\mathbb{P}^I\mathbb{P}^J}{2\beta_1\beta_2}\Big)\,, \tag{4.44}$$

where $h_a^{IJ} \equiv h^{IJ}(p_a)$. An integration measure $d\Gamma_3(p)$ is given by (3.8) in which we set $n = 3$. Now in order to fix the normalization (4.33) it suffices to compare the $\mathbb{P}^{L2}h^{RR}h^{RR}h^{LL}$ terms in (4.34) and (4.44).

To summarize we got two equivalent representations for the cubic interaction vertex given in (4.34),(4.36). The representation (4.36) being written manifestly in terms of the bosonic 'momentum' $\mathbb{P}^I$ and Grassmann 'momentum' $\Lambda$ is not convenient, however, in calculations. In contrast to this, the representation (4.34) does not show explicitly a dependence on the 'momenta' $\mathbb{P}^i$ and $\Lambda$. However the remarkable feature of representation (4.34) is that it is expressed entirely in terms of $E$- operators which have simple algebraic properties. For this reason the representation (4.34) is most convenient in various practical calculations. It is representation (4.34) that is universal and can therefore be extended in a straightforward way to the cases of $10d$ $IIA$ supergravity and SYM theories.

## 4.1 Supercharges in cubic approximation

As was said above in supersymmetric theories of interacting fields the dynamical supercharges also receive interaction - dependent corrections. In order to complete our study we have to fix dynamical supercharges too. To derive the supercharges we could use the procedure we exploited for evaluation of Hamiltonian in the preceding paragraph. For the case of supercharges there is however a shorter way, which is based on exploiting the expression for Hamiltonian density $p^-_{(3)}$ above obtained. We proceed as follows. From commutation relations

$$[P^-, Q^{-R,L}] = 0 \tag{4.45}$$

we get the following representations for the supercharge densities $q^{-R,L}$:

$$q^{-R}_{(3)} \equiv -\frac{2\hat\beta}{|\mathbb{P}|^2}Q^{-R}(\Lambda)p^-_{(3)}\,, \qquad q^{-L}_{(3)} = -\frac{2\hat\beta}{|\mathbb{P}|^2}Q^{-L}(\Lambda)p^-_{(3)}\,, \tag{4.46}$$

where we introduce operators $Q^{-R,L}(\Lambda)$ defined by

$$Q^{-R}(\Lambda) = \frac{1}{\sqrt{2}}\theta_\Lambda \not{\mathbb{P}} + \frac{1}{\hat{\beta}}\mathbb{P}^R\Lambda\,, \qquad Q^{-L}(\Lambda) \equiv \mathbb{P}^L\theta_\Lambda + \frac{1}{\sqrt{2}\hat{\beta}}\not{\mathbb{P}}\Lambda \tag{4.47}$$

and $\hat{\beta}$ is defined in (4.10). By using the representation for $p^-_{(3)}$ given in (4.36) we find from (4.46) the following manifest representation for the supercharges

$$\frac{3}{\kappa}q^{-L}_{(3)} = \frac{\mathbb{P}^L}{18\hat{\beta}}\gamma^i\Lambda\Lambda\gamma^i\Lambda - \frac{1}{36\sqrt{2}\hat{\beta}^2}\gamma^i\Lambda\Lambda\gamma^i\Lambda\Lambda\not{\mathbb{P}}\Lambda - \frac{\mathbb{P}^R}{16\cdot 63\hat{\beta}^3}\gamma^i\Lambda\Lambda\gamma^i\Lambda(\Lambda\gamma^j\Lambda)^2\,, \tag{4.48}$$

$$\frac{3}{\kappa}q^{-R}_{(3)} = -\mathbb{P}^L\Lambda + \frac{\mathbb{P}^i}{18\sqrt{2}\hat{\beta}}(8\Lambda\Lambda\gamma^i\Lambda - \gamma^{ij}\Lambda\Lambda\gamma^j\Lambda) + \frac{\mathbb{P}^R}{72\hat{\beta}^2}\Lambda(\Lambda\gamma^i\Lambda)^2\,. \tag{4.49}$$

Because $Q^{-L}$ commutes with $J^{Li}$ the supercharge density $q^{-L}_{(3)}$ admits the representation similar to (4.34) (or (4.27)):

$$q^{-L}_{(3)}(\mathbb{P},\Lambda,\beta) = \mathbb{P}^L E_q E_\rho \tilde{q}_0^{-L}(\Lambda,\beta)\,. \tag{4.50}$$

where the $\tilde{q}_0^{-L}$ can be considered to some extent as superpartner of $\tilde{V}_0$ (4.33) and is given by

$$\frac{3}{\kappa}\,\tilde{q}_0^{-L}(\Lambda,\beta) = \frac{1}{18\hat{\beta}}\gamma^i\Lambda\Lambda\gamma^i\Lambda\,. \tag{4.51}$$

Note that $E_\rho$ in (4.50) is given by (4.23), where we have to set $k=1$. The expressions for $q^{-R,L}_{(3)}$ (4.48),(4.49) can be used to rederive the $p^-_{(3)}$ (4.36) by using anticommutator $\{Q^{-R},Q^{-L}\} = -P^-$. This provides additional check to our calculations.

# 5 4-point interaction vertices

In this section we extend our analysis to 4-point interaction vertices. It should be emphasized from the very beginning that we do not consider 4-point interaction vertices of the generic $11d$ supergravity [1, 25]. Here we study 4-point vertices that are invariant with respect to linear supersymmetry transformations and do not depend on contributions of exchanges generated by cubic vertices. These 4-point vertices in their gravitational bosonic sector involve higher derivative terms that can be constructed from Riemann tensor and derivatives, i.e. they can be presented schematically as $\partial^{2n}R^4$. We will be able to find these vertices with arbitrary powers of derivatives, i.e. for arbitrary value $n$. The bosonic bodies of these supersymmetric vertices appeared in various previous studies and they are of interest because they are responsible for quantum corrections to classical action of $11d$ supergravity. Light-cone formulation we promote here allows us to find simple and compact expressions for these 4-point vertices, which might be useful in various future studies. One of interesting results of our study is that it is the supersymmetric formulation based on unconstrained light cone superfield that allows us to develop these simple and compact expressions for higher derivative vertices. Note also that a by-product of our study will be a derivation of superspace tree level 4-point scattering amplitude of the generic $11d$ supergravity theory.

Thus we are interested in 4-point Hamiltonian

$$P^-_{(4)} = \int d\Gamma_4 \Phi_{(4)} p^-_{(4)}\,, \tag{5.1}$$

where $\Phi_{(4)}$ and $d\Gamma_4$ are given by formulas (3.6),(3.7) in which we set $n=4$. According to (3.11) 4-point interaction vertex $p^-_{(4)}$ depends on the variables

$$\mathbb{P}^I_{ab}, \quad \Lambda_{ab}, \quad \beta_a\,, \qquad a,b=1,2,3,4, \tag{5.2}$$

where indices $a,b$ label four interacting fields. The 'momenta' $\mathbb{P}^I_{ab}$ and $\Lambda_{ab}$ however are not independent. Indeed making use of conservation laws it is straightforward to get the following relations

$$\mathbb{P}^I_{12} = \frac{\beta_2}{\beta_{13}}\mathbb{P}^I_{13} + \frac{\beta_1}{\beta_{13}}\mathbb{P}^I_{24}\,, \qquad \Lambda_{12} = \frac{\beta_2}{\beta_{13}}\Lambda_{13} + \frac{\beta_1}{\beta_{13}}\Lambda_{24}\,, \tag{5.3}$$

$$\beta_{ab} \equiv \beta_a + \beta_b\,. \tag{5.4}$$

Using these relations and those obtainable from them by making cyclic permutations of indices $1,2,3,4$ it is easy to see that all $\mathbb{P}^I_{ab}$ and $\Lambda_{ab}$ can be expressed in terms of 'momenta' $\mathbb{P}^I_{13}$, $\mathbb{P}^I_{24}$ and Grassmann 'momenta' $\Lambda_{13}$ $\Lambda_{24}$. Therefore we can choose the representation in which the vertex depends only on $\mathbb{P}^I_{13}$, $\mathbb{P}^I_{24}$, $\Lambda_{13}$, $\Lambda_{24}$ and $\beta_a$. The usage of $\mathbb{P}^I_{13}$, $\mathbb{P}^I_{24}$ and $\Lambda_{13}$, $\Lambda_{24}$ is advantageous because these variables transform into each other under cyclic permutations of indices $1,2,3,4$. By analogy with (4.12) we introduce then new dimensionless variables $q_{ab}$, $\rho_{ab}$:

$$q^i_{ab} \equiv \frac{\mathbb{P}^i_{ab}}{\mathbb{P}^L_{ab}}\,, \qquad \rho_{ab} \equiv \frac{\mathbb{P}^i_{ab}\mathbb{P}^i_{ab} + 2\mathbb{P}^R_{ab}\mathbb{P}^L_{ab}}{2\mathbb{P}^{L2}_{ab}}\,, \qquad \frac{\mathbb{P}^R_{ab}}{\mathbb{P}^L_{ab}} = \rho_{ab} - \frac{q^i_{ab}q^i_{ab}}{2}\,. \tag{5.5}$$

Note that we drop the summation rule for repeated four external line indices $a,b$. With this notation it is clear that the most general 4-point interaction vertex depending on $\mathbb{P}^I_{13}$, $\mathbb{P}^I_{24}$, $\Lambda_{13}$, $\Lambda_{24}$ and $\beta_a$ can be cast into the form

$$p^-_{(4)} = p^-_{(4)}(q_{13}, q_{24}, \Lambda_{13}, \Lambda_{24}, \mathbb{P}^L_{13}, \mathbb{P}^L_{24}, \rho_{13}, \rho_{24}, \beta_a)\,. \tag{5.6}$$

Thus by using (5.5) we have replaced the variables $\mathbb{P}^i_{13}$, $\mathbb{P}^R_{13}$ and $\mathbb{P}^i_{24}$, $\mathbb{P}^R_{24}$ by the respective variables $q^i_{13}$, $\rho_{13}$ and $q^i_{24}$, $\rho_{24}$.

As before the vertex $p^-_{(4)}$ can be found by exploiting commutation relations of Poincaré superalgebra and requirement of locality with respect to transverse momenta. In 4-point approximation the commutation relations between Hamiltonian $P^-$ and remaining generators $G$ take the following form in general

$$[P^-, G]_{(4)} = [P^-_{(2)}, G_{(4)}] + [P^-_{(4)}, G_{(2)}] + [P^-_{(3)}, G_{(3)}]\,. \tag{5.7}$$

Here we are interested in Hamiltonian that has no interaction corrections to cubic order i.e. we restrict our attention to the case $P^-_{(3)}=0$. This implies that in (5.7) we can drop the commutator $[P^-_{(3)}, G_{(3)}]$. Note that it is the latter commutator that is responsible for exchange contributions. The restriction $P^-_{(3)}=0$ simplifies analysis significantly. Because we have demonstrated that in cubic approximation there is only the vertex of generic $11d$ supergravity theory our analysis does not involve only the 4-point interaction vertex of that theory. In other words it is the assumption of $P^-_{(3)}=0$ that leaves aside the 4-point vertex of the generic $11d$ supergravity. Thus taking into account the restriction $P^-_{(3)}=0$ the 4-point commutator (5.7) simplifies as

$$[P^-, G]_{(4)} = [P^-_{(2)}, G_{(4)}] + [P^-_{(4)}, G_{(2)}]\,. \tag{5.8}$$

It turns out that in order to find $P^-_{(4)}$ it suffices to consider the following commutation relations $[P^-,J^{+-}]=P^-$, $[P^-,J^{IJ}]=0$ and $[P^-,J^{-L}]=0$, $[P^-,Q^{-R,L}]=0$, which lead in 4-point approximation to the equations

$$[P^-_{(4)},J^{+-}]=P^-_{(4)}\,,\qquad [P^-_{(4)},J^{IJ}]=0\,, \tag{5.9}$$

$$[P^-_{(2)},J^{-L}_{(4)}]+[P^-_{(4)},J^{-L}_{(2)}]=0\,,\qquad [P^-_{(2)},Q^{-R,L}_{(4)}]+[P^-_{(4)},Q^{-R,L}_{(2)}]=0\,. \tag{5.10}$$

Moreover we are interested in vertices $p^-_{(4)}$ whose momenta $p^I_a$ satisfy the relation

$$\sum_{a=1}^4 p_a^-=0\,,\qquad p_a^-\equiv-\frac{p_a^Ip_a^I}{\beta_a}\,. \tag{5.11}$$

Because this relation expresses simply the energy conservation law we shall refer the hypersurface defined by (5.11) to as the energy surface. Obviously the condition (5.11) decreases number of independent variables of vertex $p^-_{(4)}$ shown in r.h.s. of Eq.(5.6). Indeed making use of relation

$$\sum_{a=1}^4\frac{p_a^Ip_a^I}{\beta_a}=\frac{\mathbb{P}^I_{13}\mathbb{P}^I_{13}}{\beta_1\beta_2\beta_{13}}+\frac{\mathbb{P}^I_{24}\mathbb{P}^I_{24}}{\beta_2\beta_4\beta_{24}} \tag{5.12}$$

we are going to demonstrate that on the energy surface the variables $\rho_{13}$ and $\rho_{24}$, which were independent so far, can be now expressed in terms of remaining variables shown in r.h.s. of Eq.(5.6) and Mandelstam variable $u$

$$\rho_{13}=\frac{\beta_1\beta_3u}{2\mathbb{P}^{L2}_{13}}\,,\qquad \rho_{24}=\frac{\beta_2\beta_4u}{2\mathbb{P}^{L2}_{24}}\,. \tag{5.13}$$

To this end and in order to fix our notation we use Mandelstam variables defined by relations

$$s\equiv-(p_1+p_2)^2,\qquad t\equiv-(p_1+p_4)^2,\qquad u\equiv-(p_1+p_3)^2\,, \tag{5.14}$$

$$s+t+u=0\,. \tag{5.15}$$

Making use of on mass-shell equation for massless particles (2nd relation in (5.11)) it is easy to get a representation of Mandelstam variables in light cone basis:

$$s=\frac{\mathbb{P}^I_{12}\mathbb{P}^I_{12}}{\beta_1\beta_2}\,,\qquad t=\frac{\mathbb{P}^I_{14}\mathbb{P}^I_{14}}{\beta_1\beta_4}\,,\qquad u=\frac{\mathbb{P}^I_{13}\mathbb{P}^I_{13}}{\beta_1\beta_3}\,. \tag{5.16}$$

By using then the relations given in (5.5) and representation for $u$ given in (5.16) one can make sure that on the energy surface (5.11) (see also (5.12)) the variables $\rho_{13}$, $\rho_{24}$ admit the representation (5.13) indeed.

Now we consider commutation relations (5.9),(5.10) in which the vertex $p^-_{(4)}$ is restricted to the energy surface. Exploiting the condition (5.11) in (5.9),(5.10) leads to the following equation for 4-point Hamiltonian

$$[P^-_{(4)},J^{+-}]=P^-_{(4)}\,,\qquad [P^-_{(4)},J^{IJ}]=0\,,\qquad [P^-_{(4)},Q^{-R,L}_{(2)}]=0\,,\qquad [P^-_{(4)},J^{-L}_{(2)}]=0\,. \tag{5.17}$$

These equations in terms of the vertex $p^-_{(4)}$ defined by (5.1) take the following form

$$(\mathbb{P}^L_{13}\partial_{\mathbb{P}^L_{13}}+\mathbb{P}^L_{24}\partial_{\mathbb{P}^L_{24}}+\frac{3}{2}\Lambda_{13}\theta_{\Lambda_{13}}+\frac{3}{2}\Lambda_{24}\theta_{\Lambda_{24}}+\sum_{a=1}^4\beta_a\partial_{\beta_a}-4)p^-_{(4)}=0\,, \tag{5.18}$$

$$(J^{IJ}_{13} + J^{IJ}_{24})p^-_{(4)} = 0\,, \tag{5.19}$$

$$(Q^{-R,L}_{13} + Q^{-R,L}_{24})p^-_{(4)} = 0\,, \tag{5.20}$$

$$(J^{-L}_{13} + J^{-L}_{24})p^-_{(4)} = 0\,, \tag{5.21}$$

where the differential operators $J^{IJ}_{ab}$, $Q^{-R,L}_{ab}$ are defined by

$$J^{IJ}_{ab} \equiv \mathbb{P}^I_{ab}\partial_{\mathbb{P}^J_{ab}} - \mathbb{P}^J_{ab}\partial_{\mathbb{P}^I_{ab}} + M^{IJ}_{ab}\,, \tag{5.22}$$

$$M^{RL}_{ab} \equiv \frac{1}{2}\theta_{\Lambda_{ab}}\Lambda_{ab} - 2\,, \tag{5.23}$$

$$M^{ij}_{ab} \equiv \frac{1}{2}\theta_{\Lambda_{ab}}\gamma^{ij}\Lambda_{ab}\,, \tag{5.24}$$

$$M^{Ri}_{ab} \equiv \frac{\beta_a\beta_b\beta_{ab}}{2\sqrt{2}}\theta_{\Lambda_{ab}}\gamma^i\theta_{\Lambda_{ab}}\,, \tag{5.25}$$

$$M^{Li}_{ab} \equiv -\frac{\Lambda_{ab}\gamma^i\Lambda_{ab}}{2\sqrt{2}\beta_a\beta_b\beta_{ab}}\,, \tag{5.26}$$

$$Q^{-R}_{ab} \equiv \frac{1}{\sqrt{2}}\theta_{\Lambda_{ab}}\not{\mathbb{P}}_{ab} - \frac{1}{\beta_a\beta_b\beta_{ab}}\mathbb{P}^R_{ab}\Lambda_{ab}\,, \tag{5.27}$$

$$Q^{-L}_{ab} \equiv \mathbb{P}^L_{ab}\theta_{\Lambda_{ab}} - \frac{1}{\sqrt{2}\beta_a\beta_b\beta_{ab}}\not{\mathbb{P}}_{ab}\Lambda_{ab}\,. \tag{5.28}$$

We succeeded in finding most general solution to the defining equations (5.18)-(5.21) and our result is given by (details may be found in Appendix D):

$$p^-_{(4)} = (q^2_{_L})^2\frac{(\mathbb{P}^L_{13}\mathbb{P}^L_{24})^4}{\beta^4_{13}}E_{q_{13}}E_{q_{24}}E_u g(s,t,u)\,, \tag{5.29}$$

where operators $E_{q_{ab}}$, $E_u$ are defined to be

$$E_{q_{ab}} \equiv \exp(-q^i_{ab}M^{Li}_{ab})\,, \tag{5.30}$$

$$E_u \equiv \exp\Big(-\frac{u\Lambda^L\not{q}_{_L}\Lambda^L}{2\sqrt{2}\beta_{13}q^2_{_L}(\mathbb{P}^L_{13}\mathbb{P}^L_{24})^2}\Big)\,. \tag{5.31}$$

The new variables $q^i_{_L}$ and $\Lambda^L$ which enter the operator $E_u$ (5.31) are defined to be

$$q^i_{_L} \equiv q^i_{13} - q^i_{24}\,, \tag{5.32}$$

$$\Lambda^L \equiv \Lambda_{13}\mathbb{P}^L_{24} - \Lambda_{24}\mathbb{P}^L_{13}\,, \tag{5.33}$$

while the operators $M^{Li}_{ab}$ which enter $E_{q_{ab}}$ (5.30) are defined by (5.26).

Explicit form of a function $g(s,t,u)$ (5.29), which depends on Mandelstam variables $s$, $t$, $u$, cannot be fixed by exploiting restrictions imposed by global symmetries alone. This function is freedom of our solution. The only requirement is that the $g(s,t,u)$ be symmetric with respect to $s$, $t$, $u$[18]. If we assume that $g(s,t,u)$ admits Taylor series expansion then the lower order terms in infinite series expansion of $g(s,t,u)$ take the form

$$\begin{aligned} g(s,t,u) &= g_0 + g_2(s^2+t^2+u^2) + g_3 stu + g_4(s^4+t^4+u^4) \\ &+ g_5 stu(s^2+t^2+u^2) + g_{6;1}(s^6+t^6+u^6) + g_{6;2}(stu)^2 + \ldots . \end{aligned} \tag{5.34}$$

Note that all that is important to derive this expansion are the relation (5.15) and requirement the function $g(s,t,u)$ be symmetric in $s$, $t$, $u$. Plugging $g_n$ terms (5.34) in (5.29) gives superinvariants that consist in their gravitational bosonic body the higher derivative terms constructed from four Riemann tensors and $2n$ derivatives

$$g_n \partial^{2n} R^4 \,, \tag{5.35}$$

where $\partial^{2n}$ stands for $2n$ derivatives spread among four Riemann tensors.

Because in the literature $R^4$ terms usually are considered in covariant Lagrangian formulation we relate now the coefficient $g_0$ in expansion (5.34) with that of covariant formulation. This is to say that $g_0$ is given by

$$g_0 = \frac{8}{3}\kappa^4 \kappa_{_{(4)}} \,, \tag{5.36}$$

where $\kappa_{_{(4)}}$ is a coupling constant that appears in front of $R^4$ terms in covariant Lagrangian[19]

$$\mathcal{L}_{R^4} = \kappa_{_{(4)}} \sqrt{-g}\; W_{R^4} \,, \qquad W_{R^4} \equiv W_1 + \frac{1}{16} W_2 \,, \tag{5.37}$$

and we use the notation[20]

$$W_1 \equiv R_{42} + \frac{1}{2}R_{41} - \frac{1}{4}R_{45} - \frac{1}{8}R_{46} \,, \tag{5.38}$$

$$W_2 \equiv R_{43} + \frac{1}{2}R_{44} - 4R_{45} - 2R_{46} \,, \tag{5.39}$$

$$R_{42} = \mathrm{Tr}\;\; R_{\mu\nu}R_{\nu\rho}R_{\mu\sigma}R_{\sigma\rho} \,, \tag{5.40}$$

$$R_{41} = \mathrm{Tr}\;\; R_{\mu\nu}R_{\nu\rho}R_{\rho\sigma}R_{\sigma\mu} \,, \tag{5.41}$$

$$R_{43} = \mathrm{Tr}\;\; R_{\mu\nu}R_{\rho\sigma}\, \mathrm{Tr}\, R_{\mu\nu}R_{\rho\sigma} \,, \tag{5.42}$$

[18]Because the measure $d\Gamma_4$ and product of four superfields $\Phi_{(n)}$ in (5.1) are symmetric upon any permutations of the four external line indices 1,2,3,4, the 4-point vertex $p^{-}_{(4)}$ (5.1) should also be symmetric upon such permutations. Below we prove that $(q_{_L}^2)^2$- term which is front of function $g(s,t,u)$ (5.29) is symmetric upon any permutations of the indices 1,2,3,4 and this leads to requirement the $g(s,t,u)$ be symmetric upon any permutations of $s,t,u$.

[19]It is clear that the factor $\sqrt{-g}$ in (5.37) is not important for our analysis as we restricted ourselves to the 4-point vertices.

[20]We exploit the basis of $R^4$ terms (5.40)-(5.45), which was introduced in Appendix B2 of Ref.[26]. From that Appendix one can learn that $W_{R^4}$ admits the representation $W_{R^4} = \frac{1}{16\cdot 4!} t_8 t_8 R^4$.

$$R_{44} = (\mathrm{Tr}\ \ R_{\mu\nu}R_{\mu\nu})^2 \,, \tag{5.43}$$

$$R_{45} = \mathrm{Tr}\ \ R_{\mu\nu}R_{\mu\nu}R_{\rho\sigma}R_{\rho\sigma} \,, \tag{5.44}$$

$$R_{46} = \mathrm{Tr}\ \ R_{\mu\nu}R_{\rho\sigma}R_{\mu\nu}R_{\rho\sigma} \,. \tag{5.45}$$

In these formulas an matrix $R_{\mu\nu}$ stands for Riemann tensor $R_{\mu\nu}^{AB}$ and Tr indicates trace over Lorentz indices $A, B$.

In other words, if gravitational bosonic body of covariant supersymmetric Lagrangian is given by the expression (5.37) then the corresponding light cone gauge supersymmetric Hamiltonian is given by expressions (5.1),(5.29) with $g_0$ given by (5.36). Remaining $g_n$- terms with $n = 1, 2\ldots$ in interaction vertex (5.29) corresponding to Lagrangian (5.37) should be set equal to zero. Derivation of the relation (5.36) may be found in Appendix E.

The formula (5.36) linking constants of bosonic body of covariant Lagrangian and corresponding light cone gauge supersymmetric Hamiltonian can easily be generalized to the higher derivative terms. This is to say that if $\partial^{2n}R^4$- terms of covariant Lagrangian are given by

$$\mathcal{L}_{f\,R^4} = \sqrt{-g}\ f(s,t,u) W_{R^4} \,, \tag{5.46}$$

then the corresponding 4-point light cone gauge supersymmetric Hamiltonian is given by formulas (5.1),(5.29), where the function $g(s,t,u)$ (5.34) is expressible in terms of the function $f(s,t,u)$ as follows

$$g(s,t,u) = \frac{8}{3}\kappa^4 f(s,t,u) \,. \tag{5.47}$$

The function $f(s,t,u)$ being symmetric in $s$ $t$, $u$ has the expansion similar to that of the function $g(s,t,u)$ (5.34)[21]:

$$f(s,t,u) \;\; = \;\; f_0 + f_2(s^2+t^2+u^2) + f_3 stu + f_4(s^4+t^4+u^4) + \ldots \,. \tag{5.48}$$

A few remarks are in order.

(**i**) Rewriting the interaction vertex (5.29) in the form

$$p_{(4)}^- = Kg(s,t,u) \,, \qquad K \equiv (q_L^2)^2 \frac{(\mathbb{P}_{13}^L \mathbb{P}_{24}^L)^4}{\beta_{13}^4} E_{q_{13}} E_{q_{24}} E_u \,, \tag{5.49}$$

we remind that the quantity $K$ is usually referred to as kinematical factor. Our solution to interaction vertex implies that there is unique kinematical factor. The fact that there is unique on shell kinematical factor consisting in gravitational bosonic sector $R^4$- terms has been previously argued[22] by lifting the unique kinematical factor of $IIA$ $10d$ supergravity to eleven dimensions. Our study demonstrates this fact directly in eleven dimensions, i.e. our approach does not rely upon procedure of lifting which should be used with some care in supersymmetric theories[23]. Thus a complete set of on shell 4-point interaction vertices

[21] The coefficient $f_0$ for effective M-theory action is given in [27, 28] (see also related discussion in [29]). For the case of $11d$ supergravity the function $f(s,t,u)$ describes quantum correction to the classical action of $11d$ supergravity. Calculation of various loop corrections to the coefficients $f_0$, $f_4$, $f_6$ may be found in [30, 31]. Review of this theme and extensive list of references may be found in [32]-[34].

[22] For a discussion of covariant superfield description of $R^4$ terms in $10d$ see *e.g.* [35, 36]. For a collection papers devoted to superspace description of $R^4$ terms in $10d$, $11d$ theories see *e.g.* [37].

[23] Arguments beyond the lifting procedure may be found in [26].

is given by product of unique kinematical factor and arbitrary function $g(s,t,u)$. It is interesting to note that for $n=0,2,3,4,5$ the expansion (5.48) involves only one symmetric monomial of degree $n$ in $s,t,u$. This implies simply that for these values $n$ there is only one on shell supersymmetric 4-point vertex.

(**ii**) The expression for the vertex $p^-_{(4)}$ given in (5.29) is not manifestly polynomial in $\mathbb{P}^L_{13}$, $\mathbb{P}^L_{24}$ and $q^2_{\scriptscriptstyle L}$ because these quantities appear sometimes in denominators. One can check however that making use of various Fierz identities for gamma matrices leads to cancellation of some nonlocal expressions in $\mathbb{P}^L_{13}$, $\mathbb{P}^L_{24}$ and $q^2_{\scriptscriptstyle L}$. Remaining nonlocal expressions can be removed then by exploiting the relations

$$\mathbb{P}^R_{13} = \frac{\beta_1\beta_3}{2\mathbb{P}^L_{13}}u - \frac{\mathbb{P}^L_{13}}{2}q^2_{13}\,, \qquad \mathbb{P}^R_{24} = \frac{\beta_2\beta_4}{2\mathbb{P}^L_{24}}u - \frac{\mathbb{P}^L_{24}}{2}q^2_{24}\,, \tag{5.50}$$

which imply that some nonlocal expressions in $\mathbb{P}^L_{13}$, $\mathbb{P}^L_{24}$ can be traded for local expressions in terms of $\mathbb{P}^R_{13}$, $\mathbb{P}^R_{24}$. In fact the price we paid to get simple representation for the 4-point interaction vertex (5.29) is a loss of a manifest locality with respect to $\mathbb{P}^L_{13}$, $\mathbb{P}^L_{24}$ and $q^2_{\scriptscriptstyle L}$.

(**iii**) The interaction vertex (5.29) (or (5.49)) should be symmetric under any permutations of four external line indices 1,2,3,4. The symmetry properties of function $g(s,t,u)$ are not fixed by supersymmetries and therefore we simply demand this function be symmetric under any permutations of indices 1,2,3,4, i.e. $s,t,u$ variables. As to the kinematical factor $K$ (5.49) its form is fixed uniquely by supersymmetries and it turns out that $K$ is indeed symmetric under any permutations of 1,2,3,4. Let us demonstrate this important feature of the kinematical factor explicitly. Kinematical factor $K$ is explicitly symmetric under *cyclic* permutations of indices 1,2,3,4 and two permutations $1\leftrightarrow 3$ and $2\leftrightarrow 4$. Thus all that remains is to prove that $K$ is symmetric under permutation $2\leftrightarrow 3$. All remaining permutations can be presented as combination of the above-mentioned permutations. We proceed with analysis of $(q^2_{\scriptscriptstyle L})^2$- term that is in front of $E$-operators. This $(q^2_{\scriptscriptstyle L})^2$- term can be cast into the form that is manifestly symmetric under any permutations of 1,2,3,4. Indeed making use of relations

$$\begin{aligned} \frac{(\mathbb{P}^L_{13}\mathbb{P}^L_{24})^2}{\beta^2_{13}}q^2_{\scriptscriptstyle L} &= \mathbb{P}^L_{12}\mathbb{P}^L_{34}t - \mathbb{P}^L_{14}\mathbb{P}^L_{23}s \\ &= -\mathbb{P}^L_{13}\mathbb{P}^L_{24}s - \mathbb{P}^L_{12}\mathbb{P}^L_{34}u \\ &= \mathbb{P}^L_{13}\mathbb{P}^L_{24}t + \mathbb{P}^L_{14}\mathbb{P}^L_{23}u\,. \end{aligned} \tag{5.51}$$

it is easy to see that $(q^2_{\scriptscriptstyle L})^2$- term of $K$ can be cast into the manifestly symmetric form

$$\frac{(\mathbb{P}^L_{13}\mathbb{P}^L_{24})^4}{\beta^4_{13}}(q^2_{\scriptscriptstyle L})^2 = -(\mathbb{P}^L_{12}\mathbb{P}^L_{34})^2ut - (\mathbb{P}^L_{13}\mathbb{P}^L_{24})^2st - (\mathbb{P}^L_{14}\mathbb{P}^L_{23})^2su\,, \tag{5.52}$$

which demonstrate that $(q^2_{\scriptscriptstyle L})^2$- term of $K$ is indeed symmetric under any permutations of 1,2,3,4. All that remains is to prove that

$$E_{q_{13}}E_{q_{24}}E_u - \ \text{is invariant under permutation}\ 2\leftrightarrow 3\,. \tag{5.53}$$

This important property can be proved as follows. Introducing new 'momentum' $P^i$ by relation

$$P^i \equiv \mathbb{P}_{13}^L \mathbb{P}_{24}^L q_L^i \tag{5.54}$$

and making use of relations (5.3) one can prove that $P^i/\beta_{13}$ and $\Lambda^L/\beta_{13}$ are antisymmetric under any permutations of indices 1,2,3,4. This is to say that these quantities change sign upon permutation $2 \leftrightarrow 3$:

$$\frac{P^i}{\beta_{13}} \stackrel{2\leftrightarrow 3}{\longrightarrow} -\frac{P^i}{\beta_{13}}\,, \qquad\qquad \frac{\Lambda^L}{\beta_{13}} \stackrel{2\leftrightarrow 3}{\longrightarrow} -\frac{\Lambda^L}{\beta_{13}}\,. \tag{5.55}$$

In terms of variable $P^i$ one has the following representation for $\ln E_u$ (see (5.31)):

$$\ln E_u = -\frac{u\Lambda^L \not{P} \Lambda^L}{2\sqrt{2}\beta_{13} P^2 \mathbb{P}_{13}^L \mathbb{P}_{24}^L}\,. \tag{5.56}$$

Taking into account this representation and (5.55) we get relations

$$\ln E_u \quad \stackrel{2\leftrightarrow 3}{\longrightarrow} \quad \frac{s\Lambda^L \not{P} \Lambda^L}{2\sqrt{2}\beta_{13} P^2 \mathbb{P}_{12}^L \mathbb{P}_{34}^L} = \ln E_u - \frac{\Lambda^L \not{q}_L \Lambda^L}{2\sqrt{2}\beta_{13}^3 \mathbb{P}_{12}^L \mathbb{P}_{34}^L}\,, \tag{5.57}$$

where we use the formula

$$s = -\frac{\mathbb{P}_{12}^L \mathbb{P}_{34}^L}{\mathbb{P}_{13}^L \mathbb{P}_{24}^L} u - \frac{\mathbb{P}_{13}^L \mathbb{P}_{24}^L}{\beta_{13}^2} q_L^2\,. \tag{5.58}$$

Finally, because of relation

$$\ln E_{q_{13}} + \ln E_{q_{24}} \quad \stackrel{2\leftrightarrow 3}{\longrightarrow} \quad -q_{12}^i M_{12}^{Li} - q_{34}^i M_{34}^{Li} \tag{5.59}$$

and formula

$$-q_{12}^i M_{12}^{Li} - q_{34}^i M_{34}^{Li} - \frac{\Lambda^L \not{q}_L \Lambda^L}{2\sqrt{2}\beta_{13}^3 \mathbb{P}_{12}^L \mathbb{P}_{34}^L} = -q_{13}^i M_{13}^{Li} - q_{24}^i M_{24}^{Li}\,, \tag{5.60}$$

we see that the statement in (5.53) is indeed true.

Making use of just proved symmetry properties of the $E$- operators the 4-point vertex can be cast into more symmetric form with respect to $s$, $t$, $u$ variables. Indeed introducing the notation

$$\mathbf{J}_{ab} \equiv \mathbb{P}_{ab}^L \sqrt{E_{q_{ab}}}\,, \tag{5.61}$$

$$E_s \equiv E_u|_{2\leftrightarrow 3}\,, \qquad E_t \equiv E_u|_{3\leftrightarrow 4} \tag{5.62}$$

and using formula (5.52) the expression for the 4-point vertex (5.29) can be cast into the form

$$p_{(4)}^- = -\Big((\mathbf{J}_{12}\mathbf{J}_{34})^2 ut E_s + (\mathbf{J}_{13}\mathbf{J}_{24})^2 st E_u + (\mathbf{J}_{14}\mathbf{J}_{23})^2 us E_t\Big) g(s,t,u)\,. \tag{5.63}$$

(**iv**) The expression for the interaction vertex (5.29) (or (5.63)) can be used to obtain superspace representation for tree level 4-point scattering amplitude of the generic $11d$ supergravity theory [1]. To this end we introduce standard representation for the $S$-matrix

$$S = 1 - 2\pi \mathrm{i}\, T\,, \tag{5.64}$$

where the $T$-matrix has an expansion

$$T = \sum_n T_{(n)}\,, \tag{5.65}$$

and $n$ point interaction-dependent correction to the $T$- matrix takes standard form

$$T_{(n)} = \int d\Gamma_n\, \delta(\sum_{a=1}^n p_a^-) \prod_{a=1}^n \phi(p_a, \lambda_a)\ t_{(n)}\ . \tag{5.66}$$

In this formula $p_a^-$ are those of (5.11) and superfield $\phi(p,\lambda)$ enters a solution to free equations of motion:

$$\Phi(p,\lambda) = \exp(\mathrm{i}x^+ p^-)\phi(p,\lambda)\,. \tag{5.67}$$

$n$-point density $t_{(n)}$ in (5.66) depends on light cone momenta $\beta_a$, transverse momenta $p_a^I$ and Grassmann momenta $\lambda_a$. It is easy to check that the invariance requirement of the 4-point $T_{(4)}$-matrix with respect to $11d$ Poincaré supersymmetries leads to equations for the 4-point density $t_{(4)}$, which coincide with equations for 4-point interaction vertex $p_{(4)}^-$ restricted to the energy surface. Therefore the general solution obtained for interaction vertex (5.29) can be used to obtain solution to the $T$-matrix density $t_{(4)}$. All that remains is to find suitable function $g(s,t,u)$. The function $g(s,t,u)$ corresponding to 4-point scattering amplitude is fixed uniquely by the following two requirements:

- The scattering amplitude should has simple poles in Mandelstam variables;
- The superspace representation for the tree level 4-point scattering amplitudes being restricted to a sector of bosonic fields should has homogeneity of degree 2 in momenta $p_a^I$ and $\beta_a$.

These two requirements can be easily satisfied by choice $g(s,t,u) = -\kappa^2/(12stu)$ in (5.63). This leads to the following compact superspace representation for the tree level 4-point density $t_{(4)}$:

$$t_{(4)} = \frac{\kappa^2}{12}\Big(\frac{(\mathbf{J}_{12}\mathbf{J}_{34})^2}{s}E_s + \frac{(\mathbf{J}_{13}\mathbf{J}_{24})^2}{u}E_u + \frac{(\mathbf{J}_{14}\mathbf{J}_{23})^2}{t}E_t\Big)\,. \tag{5.68}$$

Overall coefficient is fixed so that the 4-point scattering amplitude for graviton field obtainable from (5.68) coincides with the standard graviton 4-point scattering amplitude of Einstein-Hilbert action (4.39).

Above-given expression for 4-point $T$-matrix can be used for deriving representation for 4-point scattering amplitude in various bases of in-states. For example consider representation of quantized superfield (5.67) in terms of creation and annihilation operators:

$$\phi(p,\lambda) = \frac{\epsilon(\beta)}{\sqrt{2\beta}}\,\bar a(p,\lambda) + \frac{\epsilon(-\beta)}{\sqrt{-2\beta}}\,a(-p,-\lambda)\,, \tag{5.69}$$

where $\epsilon(\beta) = 1(0)$ for $\beta > 0(\beta < 0)$ and the creation $a(p,\lambda)$ and the annihilation operators $\bar a(p,\lambda)$ satisfy the standard commutator

$$[\bar a(p,\lambda), a(p',\lambda')] = \delta^{10}(p-p')\delta^8(\lambda-\lambda')\,. \tag{5.70}$$

Introducing then the basis of ingoing superparticles $N(\beta)a(p,\lambda)|0\rangle$ we get for the tree level 4-point amplitude $\mathcal{A}_{(4)}$ defined by formula[24]

$$\langle 3,4\,|\,T_{(4)}\,|1,2\rangle = (2\pi)^{10}\delta^{11,8}\mathcal{A}_{(4)} \tag{5.71}$$

the following superspace representation

$$\mathcal{A}_{(4)} = 4!\, t_{(4)}\,, \tag{5.72}$$

where $t_{(4)}$ is given in (5.68) and we use the notation

$$\delta^{11,8} \equiv \delta^{11}(\sum_{a=1}^{4} p_a)\delta^8(\sum_{a=1}^{4}\lambda_a)\,. \tag{5.73}$$

Conventional scattering amplitude for component fields can be obtained from the above-given superspace amplitude in a straightforward way. Consider explicit representation of quantized fields in terms of polarizations which, say for graviton field, takes the form

$$h^{ij}(p) = \frac{\epsilon(\beta)}{\sqrt{2\beta}}\,\zeta^{ij}_{A}(p)\bar{a}_{A}(p) + \frac{\epsilon(-\beta)}{\sqrt{-2\beta}}\,\zeta^{ij*}_{A}(-p)a_{A}(-p)\,, \tag{5.74}$$

where $\zeta^{ij}_A$ is a basis of graviton polarization states and summation over polarization states counted by subindex $A$ is assumed. Using for the quantized $11d$ supergravity fields a representation similar to that given in (5.74) we get the conventional 4-point scattering amplitude

$$A_4 = \int \delta^8(\sum_{a=1}^{4}\lambda_a)\prod_{a=1}^{4} d^8\lambda_a \prod_{a=1}^{4}\phi_{pol}(p_a,\lambda_a)\,\mathcal{A}_4\,, \tag{5.75}$$

where a superfield $\phi_{pol}(p,\lambda)$ is obtainable from (2.16) by replacing the quantized fields by appropriate polarization vectors.

(**v**) Because the procedure of our derivation is algebraic in nature the result of this section can be easily extended to other supersymmetric theories, which have light cone formulation with manifest $so(d-4)$ transverse symmetry. In section 7 we discuss such extension to the case of $10d$ SYM theory.

# 6 Superfield form of vertex operators

In the preceding sections we have treated *field theoretical* description of $11d$ supergravity. Alternative approach to study various aspects of interacting fields is based on usage of technique of vertex operators. Because such a technique turns out to be fruitful and sometimes is preferable in some applictaions[25] we would like to reformulate our result in terms of vertex

[24] We use the normalization $N(\beta) \equiv (2\pi)^5\sqrt{2\beta}$. Basis of outgoing superparticles is defined by $\langle 0|\bar{a}(-p,-\lambda)N(-\beta)$, where $\beta<0$.

[25] For instance, world line representation for interaction vertex of particle with $11d$ supergravity in terms of components fields [38] was used to analyze loop corrections of $11d$ supergravity [38, 39]. Applications of world line approach (sometimes referred to as string inspired formalism) to discussion of UV divergences in gauge theories was discussed in [40, 41]. Review and extensive list of references may be found in [42]-[44].

operators. Thus our goal in this section is to find linearized interaction vertices of superparticle with fields of $11d$ supergravity. As before we prefer to formulate our results entirely in terms of the *unconstrained scalar superfield*[26].

In order to explain the setup we are going to use to study superparticle vertices let us start with discussion of an interaction vertex of bosonic particle with Maxwell field. In phase space approach an action of free particle takes the form

$$S_{free} = \int d\tau \mathcal{L}_{free}\,, \qquad \mathcal{L}_{free} = \mathcal{P}_\mu \dot{X}^\mu - \frac{1}{2} e \mathcal{P}_\mu \mathcal{P}^\mu\,, \tag{6.1}$$

where $X^\mu(\tau)$ and $\mathcal{P}_\mu(\tau)$ are coordinates and momenta of particle, while $e$ is 1d metric tensor of the particle world line. Phase space equations of motion take then the form

$$\dot{X}^\mu = e\mathcal{P}^\mu\,, \qquad \dot{\mathcal{P}}^\mu = 0\,, \qquad \mathcal{P}^2 = 0\,. \tag{6.2}$$

In light cone gauge

$$X^+(\tau) = \tau \tag{6.3}$$

we get solution to equations of motion

$$X^I(\tau) = x^I + \frac{p^I}{p^+}\tau\,, \qquad X^-(\tau) = x^- + \frac{p^-}{p^+}\tau\,, \tag{6.4}$$

$$\mathcal{P}^I(\tau) = p^I\,, \qquad \mathcal{P}^+(\tau) = p^+\,, \qquad \mathcal{P}^-(\tau) = p^-\,, \tag{6.5}$$

$$e(\tau) = \frac{1}{p^+}\,, \qquad p^- \equiv -\frac{p^I p^I}{2p^+}\,. \tag{6.6}$$

In covariant approach an interaction of particle with spin 1 massless field is described by action

$$S_{int} = \int d\tau\, \phi_\mu(X)\dot{X}^\mu(\tau)\,. \tag{6.7}$$

Exploiting on mass shell condition for spin 1 massless field $\phi^\mu$ taken in light cone gauge

$$\Box\phi^I = 0\,, \qquad \phi^+ = 0\,, \qquad \phi^- = -\frac{\partial_{x^I}}{\partial_{x^-}}\phi^I \tag{6.8}$$

we get the standard representation for solution to equations of motion

$$\phi^I(X^+, X) = \int \frac{d^{d-1}k}{(2\pi)^{(d-1)/2}}\, e^{\mathrm{i}k^\mu X^\mu}\, \phi^I(k)\,, \tag{6.9}$$

where on mass-shell condition in momentum representation takes the form

$$k^- = -\frac{k^I k^I}{2k^+}\,. \tag{6.10}$$

[26]Discussion of light cone world line representation for interaction vertices of particles with $11d$ supergravity in terms of *components fields* may be found in [38]. Thus as compared to this references we formulate our results entirely in terms of superfield. Also we do not exploit the widely adopted constraint $k^+ = 0$.

Plugging above given solutions of particle and field equations of motion into $S_{int}$ we get the following light cone Hamiltonian (sometimes to be referred to as interaction vertex) describing interaction of particle with spin 1 massless field:

$$P^-_{int} = \int \frac{d^{d-1}k}{(2\pi)^{(d-1)/2}}\, e^{\mathrm{i}k^\mu X^\mu} \phi^I(k) \frac{\mathbb{P}^I}{p^+k^+}\,, \tag{6.11}$$

where 'momenta' $\mathbb{P}^I$ is defined by (cf. (4.1))

$$\mathbb{P}^I \equiv p^I k^+ - k^I p^+\,. \tag{6.12}$$

Now let us turn to $11d$ supergravity. Our goal is to find of analog of $P^-_{int}$ (6.11) for the case of $11d$ superparticle interacting with $11d$ supergravity fields. To this end we could use covariant approach and find an interaction vertex by considering a particle approximation of world volume action of membrane interacting with fields of $11d$ supergravity. Such approach however is very complicated and is not useful in practical calculations because of basically the following two reasons: i) covariant $11d$ supergravity superfields involve terms of 32 powers in fermionic coordinates and have therefore very complicated structure[27]; ii) tractable quantization of Green-Schwartz superparticle action is available only in light cone gauge. Because light cone approach allows us to avoid these troubles of covariant approach it seems reasonable to use light cone approach from the very beginning. This is what we are doing below. We will start directly with light cone representation and find the interaction vertex by exploiting requirement of invariance with respect to Poincaré superalgebra.

Bosonic body of $11d$ superparticle light cone phase space[28] consists of coordinates $X^-(\tau)$, $X^I(\tau)$ and momenta $\mathcal{P}^I(\tau)$, $\mathcal{P}^+(\tau)$ given in (6.4),(6.5). Odd part of the light cone phase space of $11d$ superparticle involves eight fermionic Grassmann coordinates $\theta(\tau)$ and eight fermionic momenta $\lambda(\tau)$, which satisfy the equations of motion

$$\dot\theta(\tau) = 0\,, \qquad \dot\lambda(\tau) = 0\,. \tag{6.13}$$

Obvious solution to these equations is fixed to be

$$\theta(\tau) = \theta\,, \qquad \lambda(\tau) = \lambda\,. \tag{6.14}$$

Before going into details of deriving interaction vertex of superparticle with $11d$ supergravity fields we present our final result.

Hamiltonian describing interaction of the superparticle with the supergravity fields is found to be[29]

$$P^-_{int} = \int d\Gamma\, e^{\Omega}\, \Phi(k,\chi)\, p^-_{int}\,, \tag{6.15}$$

where $\Phi(k,\chi)$ is the light cone superfield with expansion in components fields of $11d$ supergravity given in (2.16). Measure $d\Gamma$ and quantity $\Omega$ in (6.15) are defined by formulas

$$\Omega \equiv \mathrm{i}(kx) - \chi\theta\,, \qquad (kx) \equiv k^+x^- + k^I x^I\,, \tag{6.16}$$

[27] Progress in explicit description of expansion $11d$ supergravity superfields in fermionic coordinates was achieved very recently in [45].

[28] Discussion of interrelation of this light cone superspace and covariant superspace may be found in [38].

[29] Without loss of generality we consider the interaction vertex and generators of Poincaré superalgebra for $\tau = 0$. Interaction vertex for arbitrary value of $\tau$ can be easily obtained by using solution to equations of motion for superfield (5.67) and superparticle (6.4)-(6.6),(6.14).

$$
d\Gamma = \frac{dk^+ d^9 k}{(2\pi)^5} d^8\chi \tag{6.17}
$$

and the interaction vertex $p^-_{int}$ is fixed to be

$$
\begin{aligned}
\frac{p^+}{\kappa} p^-_{int} &= \mathbb{P}^{L2} - \frac{\mathbb{P}^L}{2\sqrt{2}\hat{\beta}} \Lambda \not{\mathbb{P}} \Lambda + \frac{1}{16\hat{\beta}^2} (\Lambda \not{\mathbb{P}} \Lambda)^2 - \frac{|\mathbb{P}|^2}{9 \cdot 16\hat{\beta}^2} (\Lambda\gamma^i\Lambda)^2 \\
&+ \frac{\mathbb{P}^R}{9 \cdot 16\sqrt{2}\hat{\beta}^3} \Lambda \not{\mathbb{P}} \Lambda (\Lambda\gamma^j\Lambda)^2 + \frac{\mathbb{P}^{R2}}{2^7 \cdot 63\hat{\beta}^4} ((\Lambda\gamma^i\Lambda)^2)^2 \,,
\end{aligned} \tag{6.18}
$$

where the 'momentum' $\mathbb{P}^I$ is given in (6.12), while the quantities $\Lambda$ and $\hat{\beta}$ are defined by relations (cf. (4.2),(4.10))

$$
\Lambda \equiv \lambda k^+ - \chi p^+ \,, \tag{6.19}
$$

$$
\hat{\beta} \equiv -p^{+2} k^+ \,. \tag{6.20}
$$

In formula (6.18) $\kappa$ is gravitational constant and we choose normalization so that the interaction vertex for the graviton obtainable from (6.15) coincides with that of the standard action of particle interacting with graviton

$$
S_{int} = \int d\tau \, \frac{1}{2e} \, g_{\mu\nu}(X) \dot{X}^\mu \dot{X}^\nu \,, \tag{6.21}
$$

where the expansion for metric tensor given in (4.40) and the light cone gauge (see (4.42), (4.43), (6.3)-(6.5)) should be used.

The expression for the vertex (6.18) coincides with that of field theoretical approach given in (4.36) by module of definitions of the quantities $\mathbb{P}^I$, $\Lambda$, $\hat{\beta}$. In field theoretical vertex (4.36) we should exploit expressions for $\mathbb{P}^I$, $\Lambda$, $\hat{\beta}$ given in (4.1),(4.2),(4.10), while appropriate quantities for vertex (6.18) are defined in (6.12),(6.19),(6.20). Therefore the vertex (6.18) can be also written in terms of $E$-operators:

$$
p^-_{int} = \frac{\kappa}{p^+} \, \mathbb{P}^{L\,2} E_q E_\rho \,. \tag{6.22}
$$

As compared to (4.36) the vertex (6.22) involves an extra factor $p^+$. Appearance of this factor is related to light cone gauge on superparticle coordinate $X^+(\tau)$ given in (6.3).

## 6.1 Restrictions imposed by kinematical symmetries

As before to derive above-given linearized interaction vertex of superparticle with supergravity fields (6.18) we use the general approach of Ref.[14]. This is to say that both the particle phase variables and supergravity fields are considered to be dynamical variables and we require this dynamical system respects symmetries of $11d$ Poincaré superalgebra[30]. As before following setup of Ref.[14] we should find realization of commutation relations of Poincaré

[30] In widely used alternative approach the superparticle variables are treated as the dynamical variables, while the supergravity fields are considered as external (background) fields. In this approach variation of vertices under supersymmetry transformation of superparticle variables is replaced by supersymmetry transformation of the supergravity fields. Various application of this approach may be found in [46, 38, 47].

superalgebra for the dynamical system involving superparticle and supergravity fields. Below following this setup we demonstrate that restrictions imposed by Poincaré supersymmetries and requirement of light cone locality allows us to fix linearized interaction vertex uniquely.

Let $G_{part}$ and $G_{field}$ be respective generators of the Poincaré superalgebra for the free superparticle and supergravity fields, while $G_{int}$ be generators responsible for linearized interaction of particle with supergravity fields. The generators of the system superparticle plus supergravity are given then by

$$G \equiv G_{particle} + G_{field} + G_{int} \,. \tag{6.23}$$

The generators of free supergravity fields $G_{field}$ are given in (2.41),(2.26)-(2.39). To apply our method we need explicit expressions for free superparticle generators $G_{particle}$. These generators can be obtained in a rather straightforward way by using standard methods of the Noether procedure. We present therefore expressions for the generators without derivation[31]:

$$P^- = p^- , \qquad P^+ = p^+ \,, \qquad P^I = p^I \,, \qquad p^- = -\frac{p^I p^I}{2p^+} \tag{6.24}$$

$$J^{+I} = -\mathrm{i} x^I p^+ \,, \tag{6.25}$$

$$J^{+-} = -\mathrm{i} x^- p^+ - \frac{1}{2}\theta\lambda \,, \tag{6.26}$$

$$J^{ij} = \mathrm{i}(x^i p^j - x^j p^i) + \frac{1}{2}\theta\gamma^{ij}\lambda \,, \tag{6.27}$$

$$J^{RL} = \mathrm{i}(x^R p^L - x^L p^R) + \frac{1}{2}\theta\lambda \,, \tag{6.28}$$

$$J^{Ri} = \mathrm{i}(x^R p^i - x^i p^R) - \frac{p^+}{2\sqrt{2}}\theta\gamma^i\theta \,, \tag{6.29}$$

$$J^{Li} = \mathrm{i}(x^L p^i - x^i p^L) + \frac{1}{2\sqrt{2}p^+}\lambda\gamma^i\lambda \,, \tag{6.30}$$

$$J^{-R} = \mathrm{i}(x^- p^R - x^R p^-) - \frac{1}{2\sqrt{2}}\theta \not{p}\theta + \frac{p^R}{p^+}\theta\lambda \,, \tag{6.31}$$

$$J^{-L} = \mathrm{i}(x^- p^L - x^L p^-) + \frac{1}{2\sqrt{2}p^{+2}}\lambda \not{p}\lambda \,, \tag{6.32}$$

$$J^{-i} = \mathrm{i}(x^- p^i - x^i p^-) + \frac{1}{2p^+}\theta\gamma^i \not{p}\lambda - \frac{p^R}{2\sqrt{2}p^{+2}}\lambda\gamma^i\lambda + \frac{p^L}{2\sqrt{2}}\theta\gamma^i\theta \,, \tag{6.33}$$

$$Q^{+R} = p^+\theta \,, \tag{6.34}$$

[31] The superparticle generators are normalized so that if we plug a representation for (super)coordinates implied by (6.38) into the superparticle generators then these generators coincide with those of Section 2. The coincidence is true by module of some terms generated by ordering procedure of coordinates and momenta operators in angular generators $J^{\mu\nu}$ of Section 2.

$$Q^{+L} = \lambda\,, \tag{6.35}$$

$$Q^{-R} = \frac{1}{\sqrt{2}}\theta\, \not p + \frac{p^R}{p^+}\lambda\,, \tag{6.36}$$

$$Q^{-L} = p^L\theta + \frac{1}{\sqrt{2}p^+}\not p\lambda\,. \tag{6.37}$$

The superparticle coordinates and momenta satisfy Poisson brackets that are normalized to be[32]

$$[x^I, p^J] = \mathrm{i}\delta^{IJ}, \qquad [x^-, p^+] = \mathrm{i}\,, \qquad \{\theta, \lambda\} = 1\,. \tag{6.38}$$

Now we are going to derive restrictions imposed by the kinematical symmetries. By definition, these restrictions are obtainable from commutation relations of Hamiltonian $P^-_{int}$ with kinematical generators of $11d$ Poincaré superalgebra (2.5). To elucidate the procedure of constructing interaction vertex let us analyze the commutation relations of $P^-_{int}$ with the various kinematical generators in turn.

i) Firstly, making use of the commutation relations of $P^-_{int}$ with $P^I$, $P^+$, and $Q^{+L}$ in linearized approximation we find that a dependence of interaction vertex (6.15) on the bosonic coordinates $x^I$, $x^-$ and Grassmann coordinates $\theta$ enters throughout the quantity $\Omega$ (6.16).

ii) Secondly, we analyze restrictions imposed by commutation relations of $P^-_{int}$ with generators $J^{+I}$, $Q^{+R}$. To this end we evaluate the commutators

$$[P^-_{int}, J^{+I}] = -\int d\Gamma\, e^{\Omega}\,\Phi(k,\chi)\,\Big(k^+\partial_{k^I} + p^+\partial_{p^I}\Big)p^-_{int}\,, \tag{6.39}$$

$$[P^-_{int}, Q^{+R}] = -\int d\Gamma\, e^{\Omega}\,\Phi(k,\chi)\Big(k^+\partial_\chi + p^+\partial_\lambda\Big)p^-_{int}\,. \tag{6.40}$$

Because kinematical generators $P^I$ do not get interaction corrections commutators of $11d$ Poincaré superalgebra $[P^-, J^{+I}] = P^I$, $[P^-, Q^{+R}] = 0$ leads to commutators $[P^-_{int}, J^{+I}] = 0$, $[P^-_{int}, Q^{+R}] = 0$. The latter commutators and formulas (6.39),(6.40) give then equations

$$(k^+\partial_{k^I} + p^+\partial_{p^I})p^-_{int} = 0\,, \qquad (k^+\partial_\chi + p^+\partial_\lambda)p^-_{int} = 0\,. \tag{6.41}$$

These equations tell us that the vertex $p^-_{int}$ can be presented as

$$p^-_{int} = p^-_{int}(\mathbb{P}, \Lambda, p^+, k^+)\,. \tag{6.42}$$

In other words $p^-_{int}$ depends on $p^I$, $k^I$, $\lambda$ and $\chi$ throughout the 'momenta' $\mathbb{P}^I$ and $\Lambda$, which are defined in (6.12),(6.19).

iii) Commutator $[P^-, J^{IJ}] = 0$ gives equations

$$J^{IJ}(\mathbb{P},\Lambda)\, p^-_{int} = 0\,, \qquad J^{IJ}(\mathbb{P},\Lambda) \equiv L^{IJ}(\mathbb{P}) + M^{IJ}(\Lambda)\,, \tag{6.43}$$

where the orbital operator $L^{IJ}(\mathbb{P})$ and spin operators are given by expressions (4.5)-(4.9) in which we should use $\mathbb{P}^I$, $\Lambda$, $\hat\beta$ given in (6.12), (6.19) and (6.20) respectively.

[32]Note that in spite of appearance of imagine unity i the brackets (6.38) stand for the *classical* Poisson brackets. Such 'quantum' normalization of the classical Poisson brackets is convenient for us.

Remaining kinematical symmetry related with commutation relation of $P^-$ with $J^{+-}$ gives equation

$$(J^{+-}(\mathbb{P},\Lambda)+1)p^-_{int}=0\,, \tag{6.44}$$

where we use the notation

$$J^{+-}(\mathbb{P},\Lambda)\equiv p^+\partial_{p^+}+k^+\partial_{k^+}+\mathbb{P}^I\partial_{\mathbb{P}^I}+\frac{3}{2}\Lambda\partial_\Lambda-2\,. \tag{6.45}$$

Because Eqs.(6.43) take the same form as those given in (4.4) we can apply the same procedure we exploited while solving Eqs.(4.4). Introducing the variables $q^i$, $\rho$ as in (4.12) we get from Eqs.(6.43) the following solution

$$p^-_{int}(\mathbb{P},\Lambda,p^+,k^+)=\mathbb{P}^{L\,2}E_qE_\rho\tilde{V}_0(p^+,k^+)\,, \tag{6.46}$$

where in expressions for $E_q$ (4.20), $E_\rho$ (4.23), $M^{Li}(\Lambda)$ (4.9) we have to insert appropriate values of $\mathbb{P}^I$, $\Lambda$, $\hat{\beta}$ given in (6.12), (6.19), (6.20). In Eq.(6.46) we have extracted dimesionfull factor $\mathbb{P}^{L2}$ which is appropriate for supergravity theories.

One can check that Eq.(6.44) leads to the following equation for vertex $\tilde{V}_0(p^+,k^+)$

$$(p^+\partial_{p^+}+k^+\partial_{k^+}+1)\tilde{V}_0(p^+,k^+)=0\,. \tag{6.47}$$

Thus as in the case of field theoretical supergravity interaction vertices we see that the kinematical symmetries admit to fix dependence of particle interaction vertex on the 'momenta' $\mathbb{P}^I$ and $\Lambda$ completely, while the dependence on two light cone momenta $p^+$, $k^+$ is restricted only by one equation (6.47), i.e. to fix dependence on $p^+$, $k^+$ completely we need one additional equation. Such equation can be obtained by exploiting commutation relations between the dynamical generators and using the locality requirement. This will be done in next section.

## 6.2 Nonlinear symmetries and locality requirement

Systematic procedure of fixing dependence of interaction vertex on light cone momenta $p^+$, $k^+$ is based on study commutation relations between dynamical generators (2.6). Another reason for study of the dynamical generators is that the light cone gauge breaks manifest Lorentz symmetries and in order to check that these symmetries still present one needs to construct all generators and make sure that all commutation relations are satisfied. This is that what we are doing in this section.

The kinematical generators (2.5) do not receive interaction corrections, while the dynamical generators depend on interaction. Making use of commutation relations between the dynamical generators and kinematical generators one can make sure that interaction corrections to the dynamical generators take the following form

$$Q^{-R}_{int}=\int d\Gamma\, e^\Omega\Phi(k,\chi)q^{-R}_{int}\,, \tag{6.48}$$

$$Q^{-L}_{int}=\int d\Gamma\, e^\Omega\Phi(k,\chi)q^{-L}_{int}\,, \tag{6.49}$$

$$J^{-R}_{int}=-\mathrm{i}x^RP^-_{int}+\theta Q^{-R}_{int}+\int d\Gamma\, e^\Omega\Phi(k,\chi)j^{-R}_{int}\,, \tag{6.50}$$

$$J^{-L}_{int} = -\mathrm{i}x^L P^-_{int} + \frac{\lambda}{p^+} Q^{-L}_{int} + \int d\Gamma\, e^{\Omega} \Phi(k,\chi) j^{-L}_{int}\,, \tag{6.51}$$

$$J^{-i}_{int} = -\mathrm{i}x^i P^-_{int} + \frac{1}{\sqrt{2}}\theta\gamma^i Q^{-L}_{int} - \frac{\lambda}{\sqrt{2}\,p^+}\gamma^i Q^{-R}_{int} + \int d\Gamma\, e^{\Omega}\Phi(k,\chi) j^{-i}_{int}\,, \tag{6.52}$$

where densities $\mathcal{X} = (q^{-R}_{int}, q^{-L}_{int}, j^{-I}_{int})$ depend on 'momenta' $\mathbb{P}^I$, $\Lambda$ and light cone momenta $p^+$, $k^+$:

$$\mathcal{X} = \mathcal{X}(\mathbb{P},\Lambda; p^+, k^+)\,. \tag{6.53}$$

With expression for dynamical generators (6.48)-(6.52) at our hands we are ready to study commutators between dynamical generators. We proceed as follows.

i) Making use of commutators $[P^-, Q^{-L}] = 0$, $[P^-, Q^{-R}] = 0$ we get the following relations for the supercharge densities

$$q^{-L}_{int} = -\frac{2\hat\beta}{|\mathbb{P}|^2} Q^{-L}(\Lambda) p^-_{int}\,, \qquad q^{-R}_{int} = -\frac{2\hat\beta}{|\mathbb{P}|^2} Q^{-R}(\Lambda) p^-_{int}\,, \tag{6.54}$$

where we use the notation for differential operators:

$$Q^{-L}(\Lambda) \equiv \mathbb{P}^L \partial_\Lambda + \frac{1}{\sqrt{2}\hat\beta}\, \not{\mathbb{P}}\Lambda\,, \tag{6.55}$$

$$Q^{-R}(\Lambda) \equiv \frac{1}{\sqrt{2}}\partial_\Lambda \not{\mathbb{P}} + \frac{1}{\hat\beta}\mathbb{P}^R\Lambda\,. \tag{6.56}$$

ii) Making use of these formulas we get from the commutators $[P^-, J^{-I}] = 0$ the following expression for the density $j^{-I}_{int}$:

$$j^{-I}_{int} = -p^+\partial_{\mathbb{P}^I} p^-_{int} - \frac{2p^+\mathbb{P}^I}{|\mathbb{P}|^2}(p^+\partial_{p^+} + \Lambda\partial_\Lambda + 1)p^-_{int}\,. \tag{6.57}$$

From this formula we see that for the density $j^{-I}_{int}$ to satisfy the locality requirement we should impose on the interaction vertex the following equation

$$(p^+\partial_{p^+} + \Lambda\partial_\Lambda + 1)p^-_{int} = 0\,. \tag{6.58}$$

Taking into account the formula (6.46) we find that equation (6.58) leads to the following equation for the vertex $\tilde V_0(p^+,k^+)$:

$$(p^+\partial_{p^+} + 1)\tilde V_0(p^+,k^+) = 0\,. \tag{6.59}$$

Solution to this equation and Eq.(6.47) is fixed to be

$$\tilde V_0(p^+,k^+) = \frac{\kappa}{p^+}\,, \tag{6.60}$$

where $\kappa$ is the gravitational constant. In this formula a normalization is chosen so that to respect the normalization of the standard action of particle interacting with gravity (6.21). Note that Eq.(6.58) leads to the following simplified expression for $j^{-I}_{int}$ (6.57):

$$j^{-I}_{int} = -p^+\partial_{\mathbb{P}^I} p^-_{int}\,. \tag{6.61}$$

To complete light cone description we should write down expressions for the supercharge densities $q_{int}^{-R}$, $q_{int}^{-L}$. The expression for densities given in (6.54) are still to be formal because they are nonlocal in $\mathbb{P}^I$. However these expressions become local in transverse 'momenta' $\mathbb{P}^I$ once we insert explicit solution to interaction vertex $p_{int}^-$. Comparison with calculations of field theory densities made in Section 4.1 allows us to write down expressions for densities $q_{int}^{-R}$, $q_{int}^{-L}$ in a rather straightforward way. Indeed confronting Eqs.(6.54) with Eqs.(4.46) and formula (6.18) with (4.36) we see that $q_{int}^{-R}$, $q_{int}^{-L}$ can be obtained by making the following substitutions in formulas (4.48) (4.49): (i) in l.h.s. of expressions (4.48) (4.49) the factor $3/\kappa$ should be replaced by $p^+/\kappa$; (ii) in r.h.s. of expressions (4.48) (4.49) we should insert the expressions for $\mathbb{P}^I$, $\Lambda$, $\hat{\beta}$ given in (6.12),(6.19), (6.20) respectively.

# 7 Cubic and 4- point interaction vertices of $10d$ SYM theory

In this section we would like to extend our analysis to the case of ten dimensional SYM theory. There are two approaches to superfield light cone description of this theory. One of them keeps manifest $so(8)$ symmetries and is based on *constrained vector* superfield [9]. In alternative approach, we use below, the $so(8)$ symmetries are reduced to the manifest $so(6)$ symmetries and the action is formulated in terms of *unconstrained scalar* superfield [8]. We prefer to use the latter approach because in this approach a representation of Poincaré superalgebra generators is similar to that of $11d$ supergravity. Due to that we can straightforwardly generalize our results to the SYM case.

Our derivation for cubic and 4-point supergravity vertices was essentially algebraic because this derivation was based significantly on the usage of Poincaré superalgebra generators given in (2.26)-(2.39). Note that similar generators appear in $IIA$, $10d$ supergravity and, with some minor modification, in $10d$ SYM theory. As to $IIA$ supergravity the generalization of our results to this theory is trivial and can be achieved simply by using dimensional reduction. All that we have to do is to set one of transverse momenta equal zero in expressions for cubic vertices (4.34) (or (4.36)) and (5.29) Now let us turn to $10d$ SYM theory.

To extend our supergravity results to the case of SYM theory we should make the following modifications. First of all we have to set one of transverse momenta, say $p^7$, equal to zero. Next step is to replace odd part of light cone superspace, i.e. $\lambda$ (and $\theta$) by appropriate Grassmann variables of SYM theory. Namely, instead of $\lambda$ (and $\theta$) transforming in spinor representation of the $so(7)$ algebra we introduce $\lambda_A$ (and $\theta^A$), $A = 1, 2, 3, 4$, which transform in (anti)fundamental, i.e. vector, representation of the $su(4)$ algebra. This modification reflects the fact that $10d$ SYM theory involves 16 supercharges instead of 32 supergravity supercharges. Appropriate unconstrained light cone scalar superfield has the following expansion in powers of Grassmann momenta $\lambda_A$

$$\Psi(p,\lambda) = \beta\phi^L + \lambda_A\psi^A + \lambda_A\lambda_B\phi^{AB} + \frac{1}{\beta}(\epsilon\lambda^3)^A\psi_A - \frac{1}{\beta}(\epsilon\lambda^4)\phi^R\,, \tag{7.1}$$

where we use the notation

$$(\epsilon\lambda^3)^A \equiv \frac{1}{3!}\epsilon^{AA_1A_2A_3}\lambda_{A_1}\lambda_{A_2}\lambda_{A_3}\,, \qquad (\epsilon\lambda^4) \equiv \frac{1}{4!}\epsilon^{A_1A_2A_3A_4}\lambda_{A_1}\ldots\lambda_{A_4}\,. \tag{7.2}$$

All fields have indices of the suitable Lie algebra, which we do not show explicitly. The self-dual field $\phi^{AB}$ of the $su(4)$ algebra can be related with vector field $\phi^i$ of the $so(6)$ algebra in

a standard way

$$\phi^{AB} = \frac{1}{2\sqrt{2}}\rho^{i\,AB}\phi^i\,, \tag{7.3}$$

where normalization of the corresponding Clebsch-Gordan coefficients (or Dirac matrices[33]) is chosen so that the following relations are true

$$\rho^i_{[AB}\rho^j_{CD]} = \frac{1}{3}\epsilon_{ABCD}\delta^{ij}\,, \qquad \rho^i_{AB} = -\frac{1}{2}\epsilon_{ABCD}\rho^{i\,CD}\,. \tag{7.4}$$

Reality condition for vector field $\phi^i(-p) = \phi^{i*}(p)$ and relation (7.3) lead to self-duality condition for $\phi^{AB}$:

$$\phi^{AB}(p) = \frac{1}{2}\epsilon^{ABCD}(\phi^{CD}(-p))^*\,. \tag{7.5}$$

Fields $\phi^R$, $\phi^L$ ($\phi^L(-p) = \phi^{R*}(p)$) and $\phi^i$ describe eight bosonic physical d.o.f of SYM, while $\psi^A$ are fermionic fields subject to hermitian conjugation rule $\psi^A(-p) = (\psi_A(p))^\dagger$. Reality condition in terms of superfield $\Psi$ takes the form

$$\Psi(-p,\lambda) = \beta^2\int d\lambda^\dagger\, e^{\lambda\lambda^\dagger/\beta}(\Psi(p,\lambda))^\dagger\,. \tag{7.6}$$

Superfield $\Psi$ satisfies the following commutation relation[34]

$$[\Psi(p,\lambda),\Psi(p',\lambda')]\big|_{equal\,x^+} = \frac{1}{2\beta}\delta^9(p+p')\delta^4(\lambda+\lambda') \tag{7.7}$$

which implies that the component fields should satisfy the commutation relations

$$[\phi^I(p),\phi^J(p')] = \frac{1}{2\beta}\delta^9(p+p')\delta^{IJ}\,, \tag{7.8}$$

$$\{\psi_A(p),\psi^B(p')\} = \frac{1}{2}\delta^9(p+p')\delta_A^B\,. \tag{7.9}$$

In order to get representation of Poincaré superalgebra on the superfield $\Psi$ we should make the following replacements in expressions for generators given in (2.26)-(2.39):

(**i**) Scalar product $\theta\lambda$ in (2.28),(2.35) should be replaced by $\theta\lambda \equiv \theta^A\lambda_A$.

(**ii**) Expressions $\theta\gamma^i\theta$ and $\lambda\gamma^i\lambda$ in (2.31)-(2.37) should be replaced by $\theta\rho^i\theta \equiv \theta^A\rho^i_{AB}\theta^B$ and $\lambda\rho^i\lambda \equiv \lambda_A(\rho^i)^{AB}\lambda_B$ respectively.

(**iii**) Expression $\theta\gamma^{ij}\lambda$ in (2.29) should be replaced by $\theta^A(\rho^{ij})_A{}^B\lambda_B$, where

$$(\rho^{ij})_A{}^B \equiv \frac{1}{2}\rho^i_{AC}\rho^{j\,CB} - (i\leftrightarrow j)\,. \tag{7.10}$$

(**iv**) Last terms in (2.28),(2.30),(2.35),(2.37), *i.e.* $2$, $-2$, $-4p^R/\beta$, $-2p^i/\beta$ should be replaced by $1$, $-1$, $-2p^R/\beta$, $-p^i/\beta$ respectively.

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[33]These matrices satisfy the standard relation $\rho^{iAC}\rho^j_{CB} + (i\leftrightarrow j) = 2\delta^{ij}\delta^A_B$ and we adopt complex conjugation rule $\rho^{iAB*} = -\rho^i_{AB}$. Note that our sign convention in (7.4) differs from that adopted in formulas (A.7), (A.9) in Ref.[8]. In this section we use $so(6)$ vector indices $i,j = 1,\ldots,6$, $so(8)$ vector indices $I,J = 1,\ldots,6,R,L$ and $su(4)$ vector indices $A,B = 1,\ldots,4$.

[34]Grassmann delta function is chosen to be $\delta^4(\lambda) \equiv (\epsilon\lambda^4)$ (see (7.2)). Accordingly an integral over Garssman variables is normalized to be $\int d^4\lambda\delta^4(\lambda) = 1$.

After this we can extend our supergravity calculations to the case of SYM theory in a rather straightforward way. We note that it is usage of $E$-operators that allows us to do such straightforward extension. Below we present interaction vertices of SYM theory without derivation. Let us first consider the cubic vertices.

The superspace representation of $10d$ SYM theory Hamiltonian in cubic approximation is given by

$$P_{(3)}^- = \int d\Gamma_3 \mathrm{Tr}\, (\prod_{a=1}^3 \Psi(p_a,\lambda_a)) p_{(3)}^-(\mathbb{P},\Lambda,\beta)\,, \tag{7.11}$$

where the $d\Gamma_3$ is obtainable from (3.7)-(3.9) by setting $n=3$, $d=10$ and replacing $d^8\lambda$ by $d^4\lambda$. The cubic vertex is found to be[35]

$$p_{(3)}^-(\mathbb{P},\Lambda,\beta) = -\frac{2g_{_{YM}}}{3}\,\mathbb{P}^L E_q E_\rho\,, \tag{7.12}$$

where operators $E_q$ and $E_\rho$ are given by[36]

$$E_q = \exp(-q^j M^{Lj}(\Lambda))\,, \tag{7.13}$$

$$E_\rho \equiv 1 - \frac{\rho}{6} M^{Lj}(\Lambda) M^{Lj}(\Lambda)\,, \tag{7.14}$$

$$M^{Li}(\Lambda) \equiv \frac{1}{2\sqrt{2}\hat\beta}\Lambda\rho^i\Lambda\,. \tag{7.15}$$

The quantity $\hat\beta$ is given by (4.10), while $\Lambda$ is obtainable from (4.2) replacing the $so(7)$ Grassmann momentum $\lambda_\alpha$ by the $su(4)$ Grassmann momentum $\lambda_A$. The expression for interaction vertex can be re-written manifestly in terms of momenta $\mathbb{P}^I$ and $\Lambda$:

$$p_{(3)}^-(\mathbb{P},\Lambda,\beta) = -\frac{2g_{_{YM}}}{3}\Big(\mathbb{P}^L - \frac{\mathbb{P}^i}{2\sqrt{2}\hat\beta}\Lambda\rho^i\Lambda - \frac{\mathbb{P}^R}{\hat\beta^2}(\epsilon\Lambda^4)\Big)\,. \tag{7.16}$$

Normalization coefficient in the vertex (7.12) (or (7.16)) is chosen so that the bosonic body of covariant action corresponding to the supersymmetric light cone Hamiltonian (7.11) is given by

$$S_{_{YM}} = \int d^{10}x\, \mathcal{L}_{_{YM}}\,, \qquad \mathcal{L}_{_{YM}} = \mathrm{Tr}\, -\frac{1}{4}F_{\mu\nu}F_{\mu\nu}\,, \tag{7.17}$$

where field strength is defined as

$$F_{\mu\nu} = \partial_\mu\phi_\nu - \partial_\nu\phi_\mu + \mathrm{i}g_{_{YM}}[\phi_\mu,\phi_\nu] \tag{7.18}$$

and gauge field $\phi_\mu$ is a hermitian matrix $\phi_\mu^\dagger = \phi_\mu$.

Now let us consider higher derivative 4-point vertices. We are interested in superspace light cone representation of 4-point interaction vertices involving in their bosonic sector $\partial^n F^4$ terms[37]. Following step by step the procedure we used while deriving supergravity 4-point

[35] The representation for the cubic vertex in terms of $E$-operators (7.12) is result of this paper. Explicit representation in terms of 'momenta' $\mathbb{P}^I$ and $\Lambda$ (see Eq.(7.16) below) was found in Ref.[8].

[36] Operator $E_\rho$ can be obtained by setting $d=10$ and $k=1$ in the general solution given in (B.16).

[37] Supersymmetric completion of *non-abelian* $F^4$ terms to the second order in fermions was obtained in [48]. Full supersymmetric completion of *abelian* and *non-abelian* $F^4$ terms was obtained in [49] and [50] respectively. Supersymmetric action to all orders in *abelian* field strength $F$ and fermions was found in [51].

vertex we obtain the following 4-point supersymmetric Hamiltonian:

$$P^-_{(4)} = \int d\Gamma_4 \mathrm{Tr}\,(\prod_{a=1}^4 \Psi(p_a,\lambda_a))p^-_{(4)}(\mathbb{P},\Lambda,\beta)\,, \tag{7.19}$$

where the expression for 4-point vertex is fixed to be

$$p^-_{(4)} = q^2_{{}_L}\frac{(\mathbb{P}^L_{13}\mathbb{P}^L_{24})^2}{\beta^2_{13}}\,E_{q_{13}}E_{q_{24}}E_u g(s,t)\,, \tag{7.20}$$

and the measure $d\Gamma_4$ is obtainable from (3.7) by setting $n=4$, $d=10$ and replacing $d^8\lambda$ by $d^4\lambda$. The variables $q^i_{{}_L}$ and $\Lambda^L$ take the same form as in (5.32),(5.33), while the operators $E_{q_{ab}}$ and $E_u$ are given by Eqs.(5.30),(5.31) in which we have to make replacement above mentioned in (ii) and (iii). Function $g(s,t)$ depending on Mandelstam variables $s$, $t$ cannot be fixed by global supersymmetries. This function should be cyclically symmetric

$$g(s,t) = g(t,s) \tag{7.21}$$

and if we assume that the function $g(s,t)$ admits Taylor series expansion then lower order terms of expansion take the form

$$g(s,t) = g_0 + g_1 u + g_{2;1}st + g_{2;2}u^2 + \ldots\,. \tag{7.22}$$

As compared to analogous expressions for $11d$ supergravity vertex (5.29) we see that $q^2_{{}_L}$- term which is in front of the $E$-operators (7.20) turns out to be a square root of the corresponding term in the supergravity vertex (5.29). This can be considered to some extent as a sort of Kawai Lewellen Tye relationship between gravity and gauge theory.

Making use of formulas (5.51) it is easy to see that $q^2_{{}_L}$- term in (7.20) is symmetric upon any permutations of the four external line indices 1,2,3,4. The product of $E$-operators in (7.20) is also symmetric upon such permutations[38]. Making use of these symmetry properties the 4-point vertex can be cast into more symmetric form with respect to $s$, $t$ variables. This is to say that introducing new quantities $\mathbf{J}_{ab}$:

$$\mathbf{J}_{ab} \equiv \mathbb{P}^L_{ab}E_{q_{ab}}\,, \tag{7.23}$$

and exploiting the first relation in (5.51) we can cast the expression (7.20) into the form

$$p^-_{(4)} \;=\; (\mathbf{J}_{12}\mathbf{J}_{34}tE_s - \mathbf{J}_{14}\mathbf{J}_{23}sE_t)\,g(s,t) \tag{7.24}$$

where operators $E_s$ and $E_t$ are obtainable from $E_u$ in the same way as in (5.62).

The function $g(s,t)$ can be expressed in terms of constants that appear in covariant Lagrangian formulation. Let a bosonic body of covariant supersymmetric Lagrangian at order $F^4$ is given by[39]

$$\mathcal{L}_{{}_{F^4}} \;=\; g_{{}_{F^4}}W_{{}_{F^4}}\,, \tag{7.25}$$

$$\begin{aligned} W_{{}_{F^4}} \;&\equiv\; \mathrm{Tr}\; F_{\mu\rho}F_{\nu\rho}F_{\mu\sigma}F_{\nu\sigma} + \frac{1}{2}F_{\mu\rho}F_{\nu\rho}F_{\nu\sigma}F_{\mu\sigma} \\ &-\; \frac{1}{4}F_{\mu\nu}F_{\mu\nu}F_{\rho\sigma}F_{\rho\sigma} - \frac{1}{8}F_{\mu\nu}F_{\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,, \end{aligned} \tag{7.26}$$

[38]This total symmetry of the $q^2_{{}_L}$- term and product of the $E$-operators is related to the well known total symmetry of the kinematic factor $K$ that enters scattering amplitude of type $I$ superstring theory (see *e.g.* section 4.2. in Ref.[52]).

[39]Lagrangian (7.25) describes $F^4$ corrections to low energy dynamics of massless spin 1 modes of type I superstring theory [53, 54].

where $g_{_{F^4}}$ is some constant. Then the corresponding light cone gauge supersymmetric Hamiltonian is given by Eqs.(7.19),(7.20), where the function $g(s,t)$ (7.22) is fixed to be

$$g(s,t) = \frac{1}{2} g_{_{F^4}} \,. \tag{7.27}$$

Details of derivation this relationship may be found at the end of Appendix E. The overall constant $g_{_{F^4}}$ in (7.25) is dynamical-dependent and cannot be fixed by global symmetries. For the case of tree level superstring effective Lagrangian for massless spin 1 fields this constant can be expressed in terms of string tension [53, 54].

The formula (5.36) linking constants of bosonic body of covariant Lagrangian and corresponding light cone gauge supersymmetric Hamiltonian can easily be generalized to the higher derivative $\partial^n F^4$ terms. Namely, if bosonic body of covariant Lagrangian is given by

$$\mathcal{L}_{_{f\,F^4}} = f(s,t) W_{_{F^4}} \,, \tag{7.28}$$

$$f(s,t) = f_0 + f_1 u + f_{2;1} st + f_{2;2} u^2 + \ldots \,, \tag{7.29}$$

then the corresponding light cone gauge supersymmetric Hamiltonian is given by Eqs.(7.19), (7.20), where the function $g(s,t)$ (7.22) is fixed to be[40]

$$g(s,t) = \frac{1}{2} f(s,t) \,. \tag{7.30}$$

Concrete form of the function $f(s,t)$ (and hence $g(s,t)$) is dynamical - dependent. For the case of superstring theory the coefficients $f_0, f_1, \ldots$ describe tree level effective Lagrangian for massless modes as well as quantum string loop corrections. For $10d$ SYM theory these coefficients are responsible for quantum loop corrections. Evaluation of contribution of the SYM theory one-loop UV divergencies to the coefficients $f_0$, $f_1$ may be found in [40] (the case of constant abelian $F$ was considered in [55]).

As in the case of $11d$ supergravity the expression for interaction vertex (5.29) can be used to obtain superspace representation for tree level 4-point scattering amplitude of the generic $10d$ SYM theory. As before to get 4-point $T$- matrix from $P^-_{(4)}$ (7.19) we should multiply the 4-point interaction vertex $p^-_{(4)}$ (7.20) by delta function $\delta(\sum_{a=1}^4 p^-_a)$ that respects energy conservation law. Beyond this the function $g(s,t)$ should be chosen so that to respect the following two requirements: (i) the scattering amplitude should has simple poles in Mandelstam variables. (ii) the amplitude being restricted to the sector of bosonic fields should has homogeneity of degree 0 in momenta $p^I_a$, $\beta_a$. These two requirements can be fulfilled by the following choice of $g(s,t)$:

$$g(s,t) = \frac{g^2_{_{YM}}}{st} \,. \tag{7.31}$$

This leads to the following compact superspace representation for tree level 4-point $T$-matrix of $10d$ SYM theory:

$$T_{(4)} = \int d\Gamma_4 \, \delta(\sum_{a=1}^4 p^-_a) \, \mathrm{Tr} \prod_{a=1}^4 \psi(p_a,\lambda_a) \; t_{(4)} \,. \tag{7.32}$$

[40]Note that for establishing a relation (7.30) we do not need concrete form of expansion for the function $f(s,t)$ given in (7.29). All that is required for derivation the relation (7.30) is 'crossing symmetry' of the function $f$: $f(s,t) = f(t,s)$.

where the expression for density $t_{(4)}$ is given by

$$t_{(4)} = g_{YM}^2 \left( \frac{\mathbf{J}_{12}\mathbf{J}_{34}}{s} E_s - \frac{\mathbf{J}_{14}\mathbf{J}_{23}}{t} E_t \right) \tag{7.33}$$

and a superfield $\psi(p,\lambda)$ enters a solution to free equations of motion:

$$\Psi(p,\lambda) = \exp(\mathrm{i}x^+ p^-)\psi(p,\lambda)\,. \tag{7.34}$$

We introduce then the standard decomposition similar to that given in (5.69)

$$\psi(p,\lambda) = \frac{\epsilon(\beta)}{\sqrt{2\beta}}\,\sigma\bar{a}(p,\lambda) + \frac{\epsilon(-\beta)}{\sqrt{-2\beta}}\,\sigma a(-p,-\lambda)\,, \tag{7.35}$$

where $\sigma$ are Lie algebra matrices and we define a superspace 4-point amplitude by relation

$$\langle 3,4\,|\,T_{(4)}\,|1,2\rangle = (2\pi)^9 \delta^{10,4} \mathcal{A}_{(4)} \tag{7.36}$$

in which $\delta^{10,4}$-function is an analog of (5.73). These relations lead to the following 4-point amplitude

$$\mathcal{A}_4 = \mathcal{A}_4^{abel}\mathrm{Tr}\,(\sigma_1 \ldots \sigma_4) + \text{ non-cyclic perms. of } 1,2,3,4\,, \tag{7.37}$$

where we introduce 'abelian' part of the amplitude defined by

$$\mathcal{A}_4^{abel} = 4t_{(4)}\,. \tag{7.38}$$

Superspace light cone representation for 4-point scattering amplitude of SYM theory was obtained in [8] (see formula (5.48) in Ref.[8]). Attractive feature of our alternative representation for 4-point scattering amplitude (7.37),(7.38) is that the complicated dependence of the amplitude on the bosonic 'momenta' $\mathbb{P}_{ab}^i$ and the Grassmann 'momenta' $\Lambda_{ab}$ is entirely collected in the $E$-operators. It is usage of the $E$- operators that allows us to find compact representation for amplitude given in (7.33),(7.37).

# 8 Conclusions

We developed the superspace light cone formalism for $11d$ supergravity. In this paper we applied this formalism to study of the superspace representation of the higher derivative 4-point interaction vertices. We found also superfield representation for cubic interaction vertex and for the tree level 4-point scattering amplitude of the generic $11d$ supergravity [1]. By analogy with the fact that it is light cone gauge cubic vertices of the $10d$ supergravity theories that admit natural extension to superstring field theories we expect that our vertices will admit natural extension to M-theory. Because the formalism we presented is algebraic in nature it allows us to find various interaction vertices in a relative straightforward way. Comparison of this formalism with other approaches available in the literature leads us to the conclusion that this is a very efficient formalism.

Long term motivation for our study is related to conjectured supergravity theory in $AdS_{11}$ spacetime [56]. As is well known the standard $11d$ supergravity [1] does not admit an extension with a cosmological constant, i.e. does not have $AdS_{11}$ vacuum [57, 58](see also [59, 60]).

On the one hand, in Ref.[56] certain massless $AdS_{11}$ graviton supermultiplet was found[41]. This novel supermultiplet contains fields of the usual $11d$ supergravity plus additional ones. One can expect that these additional fields may allow one to overcome no-go theorem and construct a consistent supergravity admitting $AdS_{11}$ ground state. Certain massless $AdS_{11}$ graviton multiplet is also predicted by eleven dimensional version of $AdS_{10}$ higher spin gauge theories discovered in [64]. These theories allow more or less straightforward generalization to $AdS_{11}$. Usually a tower of infinite higher spin fields contains of supergravity multiplet and therefore one expects that an extension to $d = 11$ of ten dimensional theories discussed in [65, 66] also describes some $AdS_{11}$ graviton multiplet. On the other hand in [67] it was demonstrated that, under certain assumption about *spontaneous breaking of AdS symmetries*, totally symmetric massless higher spin field in $AdS_{d+1}$ spacetime leads to a massive field in $d$ dimensional Minkowski spacetime whose mass and spin are related in the same manner as for a massive field belonging to the leading Regge trajectory of string theory. *This suggests that superstrings in 10 dimensional Minkowski space-time, viewed as a boundary, could be related to higher spin massless fields theory living in $AdS_{11}$ spacetime, viewed as a bulk*. Because superstring theory admits simple and elegant formulation in light cone gauge one can expect that eleven dimensional theories should also be formulated within the light cone gauge. In this perspective the study of this paper can be considered as a warm-up for generalization to $AdS_{11}$ spacetime. Light-cone form of field dynamics in $AdS$ spacetime developed in [68] implies that such generalization is possible in principle.

The results presented here should have a number of interesting applications and generalizations, some of which are:

(i) generalization to eleven dimensional anti-de Sitter spacetime $AdS_{11}$ and study of interaction vertices for massless $AdS_{11}$ graviton supermultiplet found in [56].

(ii) generalization to 3- and 4- point interaction vertices of type IIB supergravity in $AdS_5\times S^5$ background and application to study of superspace form of AdS/CFT correspondence at the level of 3- and 4-points correlation functions.

(iii) application of manifestly supersymmetric light-cone formalism to the study of the various aspects of M-theory along the lines [69, 70].

Another interesting application, which triggered our investigation, is related to certain massless (nonsupersymmetric) triplets in $d = 11$, the dimension of M-theory. It was found in [71] that some irreps of $so(9)$ algebra naturally group together into triplets to be referred to as Euler triplets which are such that bosonic and fermionic degrees of freedom match up the same way as in $11d$ supergravity. Later on Euler triplets were studied in Refs.[72]-[74] and it was conjectured that these triplets might be organized in a relativistic theory so that this theory would presumably be finite. The methods we developed and used in this paper for study $11d$ supergravity admit straightforward generalization to study of higher spin Euler triplets. On the other hand a world line approach used in Refs.[38, 39] for evaluation of quantum loop corrections to $11d$ supergravity can also be generalized to study loop corrections for higher spin fields in a relative straightforward way. In principle just mentioned methods and approaches should make it possible to address question of UV finiteness of higher spin fields theory based on Euler triplets. We hope to return to these problems in future publications.

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[41] This novel graviton supermultiplet is to transform in representation of orthosymplectic superalgebra $osp(32,1)$. Recent interesting discussion of unitary representations of $osp(32,1)$ superalgebra and gravity theories based on such superalgebra may be found in [61] and [62, 63] respectively.

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# Appendix A $11d$ Poincaré superalgebra and gravitino field in $so(7)$ basis

In order to cast $11d$ Poincaré superalgebra commutators (2.2) into the $so(7)$ basis we first transform them to the $so(9)$ basis. To this end we use the following decomposition of $32\times 32$ gamma matrices and charge conjugation matrix

$$\gamma_{32}^{\mu} = \left(\begin{array}{cc} \delta_I^\mu \gamma_{16}^I & \sqrt{2}\delta_+^\mu \\ \sqrt{2}\delta_-^\mu & -\delta_I^\mu \gamma_{16}^I \end{array}\right), \qquad C_{32} = \left(\begin{array}{cc} 0 & C_{16} \\ -C_{16} & 0 \end{array}\right), \tag{A.1}$$

$\{\gamma_{32}^\mu, \gamma_{32}^\nu\} = 2\eta^{\mu\nu}$, $\gamma_{32}^{\mu\,t} = -C_{32}\gamma_{32}^\mu C_{32}^{-1}$ and $\eta^{\mu\nu}$ is mostly positive flat metric tensor. $\gamma_{16}^I$ and $C_{16}$ are $16\times 16$ gamma matrices and charge conjugation matrix respectively:

$$\{\gamma_{16}^I, \gamma_{16}^J\} = 2\delta^{IJ}\,, \qquad \gamma_{16}^{I\,t} = C_{16}\gamma_{16}^I C_{16}^{-1}\,, \qquad C_{16}^t = C_{16}\,. \tag{A.2}$$

Decomposition of the 32-component supercharges $Q$ into two 16-component supercharges $Q^\pm$ and using Majorana condition $Q^\dagger\gamma_{32}^0 = Q^t C_{32}$ gives

$$Q = 2^{1/4}\left(\begin{array}{c} Q^- \\ Q^+ \end{array}\right), \qquad (Q^\pm)^\dagger = (Q^\pm)^t C_{16}\,. \tag{A.3}$$

In the $so(9)$ basis (anti)commutators of Poincaré superalgebra given in (2.2) take then the form

$$\{Q^\pm, Q^\pm\} = \pm C_{16}P^\pm\,, \qquad \{Q^+, Q^-\} = \frac{1}{\sqrt{2}}\gamma_{16}^I C_{16} P^I\,, \tag{A.4}$$

$$[J^{+-}, Q^\pm] = \pm\frac{1}{2}Q^\pm, \qquad [J^{IJ}, Q^\pm] = -\frac{1}{2}\gamma_{16}^{IJ}Q^\pm, \qquad [J^{\pm I}, Q^\mp] = \pm\frac{1}{\sqrt{2}}\gamma_{16}^I Q^\pm. \tag{A.5}$$

To convert these commutators into those of $so(7)$ basis we use the following decomposition of $\gamma_{16}^I$ and $C_{16}$ matrices

$$\gamma_{16}^I = \left(\begin{array}{cc} \delta_i^I\gamma^i & \sqrt{2}\delta_R^I \\ \sqrt{2}\delta_L^I & -\delta_i^I\gamma^i \end{array}\right), \qquad C_{16} = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right), \tag{A.6}$$

where $\gamma^i$ are $8\times 8$ gamma matrices that are antisymmetric and hermitian: $\gamma^{it} = -\gamma^i$, $\gamma^{i\dagger} = \gamma^i$. The decomposition for supercharges $Q^\pm$ we use is

$$Q^\pm = 2^{1/4}\left(\begin{array}{c} Q^{\pm L} \\ Q^{\pm R} \end{array}\right). \tag{A.7}$$

In terms of the supercharges $Q^{\pm R,L}$ the Majorana condition (A.3) takes the form $Q^{\pm R\dagger} = Q^{\pm L}$. By exploiting these formulas in (anti)commutators of the Poincaré superalgebra taken

in the $so(9)$ basis (A.4),(A.5) we arrive at the $so(7)$ basis (anti)commutators given in Section 2. The reader interested in the $so(7)$ $\gamma^i$- matrices identities is advised to consult the appendices of Ref.[75].

Now let us describe relationship of the $so(7)$ basis physical components of gravitino field, which enter superfield expansion (2.16), with the nomenclature of covariant approach. Lorentz covariant equations of motion and constraints for gravitino field take the form

$$\gamma_{32}^\mu p_\mu \psi_\nu = 0\,, \qquad p^\mu \psi_\mu = 0\,, \qquad \gamma_{32}^\mu \psi_\mu = 0 \tag{A.8}$$

Making use of gauge $\psi_- = 0$, one proves that the components given by $\psi_I^\oplus \equiv \frac{1}{2}\gamma_{32}^-\gamma_{32}^+\psi_I$ are physical fields, while the components $\psi_I^\ominus \equiv \frac{1}{2}\gamma_{32}^+\gamma_{32}^-\psi_I$ are non-physical d.o.f. The physical gravitino field satisfies algebraic constraint which by exploiting the 16-component notation can be written in the $so(9)$ basis as

$$\gamma_{16}^I \psi_I^\oplus = 0\,. \tag{A.9}$$

We solve this constraint by exploiting $so(7)$ basis. Namely, introducing

$$\psi_I^{\oplus\,R} \equiv \frac{1}{2}\gamma_{16}^L\gamma_{16}^R\psi_I^\oplus\,, \qquad \psi_I^{\oplus\,L} \equiv \frac{1}{2}\gamma_{16}^R\gamma_{16}^L\psi_I^\oplus\,, \tag{A.10}$$

it is easily seen that the components $\psi_i^{\oplus\,R}$, $\psi_L^{\oplus\,R}$ (and their hermitian conjugated partners $\psi_i^{\oplus\,L}$, $\psi_R^{\oplus\,L}$) are independent, while the remaining components $\psi_R^{\oplus\,R}$ (and $\psi_L^{\oplus\,L}$) are expressible in terms of that independent components

$$\psi_R^{+R} = -\frac{1}{\sqrt{2}}\gamma^i\psi_i^{+L}\,, \qquad \psi_L^{+L} = \frac{1}{\sqrt{2}}\gamma^i\psi_i^{+R}\,. \tag{A.11}$$

It is the components $\psi_i^{\oplus\,R}$, $\psi_L^{\oplus\,R}$ (and $\psi_i^{\oplus\,L}$, $\psi_R^{\oplus\,L}$) that enter expansion of the superfield $\Phi$ in (2.16).

# Appendix B Derivation of representation for the cubic vertex (4.22).

To derive the representation (4.22) we use $RL$, $Ri$ and $ij$ parts of Eqs.(4.4). Acting with angular momenta $J^{IJ}(\mathbb{P},\Lambda)$ on the vertex $p_{(3)}^-$ (4.21) we find the expressions

$$J^{RL}(\mathbb{P},\Lambda)p_{(3)}^- = (\mathbb{P}^L)^k E_q\Big(M^{RL}(\Lambda) + 2\rho\partial_\rho - k\Big)\tilde{V}\,, \tag{B.1}$$

$$J^{Ri}(\mathbb{P},\Lambda)p_{(3)}^- = q^i J^{RL}(\Lambda)p_{(3)}^- + (\mathbb{P}^L)^k E_q\Big(M^{Ri}(\Lambda) - \rho M^{Li}(\Lambda) + q^j M^{ij}(\Lambda)\Big)\tilde{V}\,, \tag{B.2}$$

$$J^{ij}(\mathbb{P},\Lambda)p_{(3)}^- = (\mathbb{P}^L)^k E_q M^{ij}(\Lambda)\tilde{V}\,. \tag{B.3}$$

From these expressions it is easily seen that the $RL$, $Ri$ and $ij$ parts of Eqs.(4.4) lead to the following equations for $\tilde{V}$

$$(M^{RL}(\Lambda) + 2\rho\partial_\rho - k)\tilde{V} = 0\,, \tag{B.4}$$

$$(M^{Ri}(\Lambda) - \rho M^{Li}(\Lambda))\tilde{V} = 0\,, \tag{B.5}$$

$$M^{ij}(\Lambda)\tilde{V} = 0\,. \tag{B.6}$$

From the relations in (4.12) and the fact that the vertex $p^-_{(3)}$ is a monomial of degree $k$ in $\mathbb{P}^I$ it follows that $\tilde{V}$ should be polynomial of degree $k$ in $\rho$, *i.e.* we can use the expansion

$$\tilde{V}(\rho,\Lambda,\beta) = \sum_{n=0}^{k} \rho^n \tilde{V}_n(\Lambda,\beta)\,. \tag{B.7}$$

Plugging this expansion in Eq.(B.5) we get the following equations[42]

$$M^{Ri}(\Lambda)\tilde{V}_n = M^{Li}(\Lambda)\tilde{V}_{n-1}\,, \qquad n = 1,\ldots,k\,, \tag{B.8}$$

$$M^{Ri}(\Lambda)\tilde{V}_0 = 0\,, \tag{B.9}$$

while Eqs.(B.4),(B.6) lead to the respective equations for $\tilde{V}_0$ given in (4.24),(4.26).

Now we focus on Eqs.(B.8). These equations tell us that $\tilde{V}_n$ can be expressed in terms of $\tilde{V}_0$. Making use of (B.8) and (B.6) one can make sure that $\tilde{V}_n$ can be presented in the form

$$\tilde{V}_n = f_n (M^{Lj}(\Lambda)M^{Lj}(\Lambda))^n \tilde{V}_0\,, \tag{B.10}$$

which should be supplemented by obvious initial condition $f_0 = 1$. Now making use of Eqs.(4.24)-(4.26) and commutation relations

$$[M^{Ri},\,(M^{Lj}M^{Lj})^n] = (M^{Lj}M^{Lj})^{n-1} 2n (M^{Lj}M^{ji} - M^{Li}M^{RL} - (\frac{N'}{2} - n)M^{Li}) \tag{B.11}$$

we get

$$M^{Ri}(M^{Lj}M^{Lj})^n \tilde{V}_0 = -n(N'+2k-2n)(M^{Lj}M^{Lj})^{n-1}M^{Li}\tilde{V}_0\,, \tag{B.12}$$

$N' \equiv d-4$, where we use Eq.(B.6) and for flexibility we keep the spacetime dimension $d$ to be arbitrary. Making use of (B.12) and (B.10) in (B.8) gives the following equations for $f_n$

$$\frac{f_{n-1}}{f_n} = -n(N'+2k-2n)\,. \tag{B.13}$$

Solution to these equations with $f_0 = 1$ is easily found to be

$$f_n = (-)^n \frac{\Gamma(\frac{N'}{2}+k-n)}{2^n n! \Gamma(\frac{N'}{2}+k)}\,, \tag{B.14}$$

where $\Gamma$ is the Euler gamma function. Collecting all steps of derivation we arrive at solution

$$\tilde{V}(\rho,\Lambda,\beta) = E_\rho \tilde{V}_0(\Lambda,\beta)\,, \tag{B.15}$$

$$E_\rho \equiv \sum_{n=0}^{k} (-\rho)^n \frac{\Gamma(\frac{d-4}{2}+k-n)}{2^n n! \Gamma(\frac{d-4}{2}+k)} (M^{Lj}(\Lambda)M^{Lj}(\Lambda))^n\,. \tag{B.16}$$

Restriction to eleven dimensions $d = 11$ leads to the desired relation (4.22).

[42] In addition to Eqs.(B.8),(B.9) one has extra equation $M^{Li}(\Lambda)\tilde{V}_k = 0$. Because $\tilde{V}_k$ turns out to be monomial of degree $4k$ in Grassmann momentum $\Lambda$ (note that $\Lambda^9 = 0$ as $\Lambda$ has eight components) this extra equation amounts to the equation $\Lambda^{4k+2} = 0$ which satisfies automatically for supergravity theories $k \geq 2$.

# Appendix C Derivation of expression for density $j^{-I}$ (4.31)

In this appendix we outline procedure of deriving the expression for density $j^{-I}$ given in (4.31). To simplify presentation we focus on the calculation of $j^{-L}_{(3)}$. For flexibility we start with calculation of $n$-point commutators for arbitrary value of $n$:

$$[P^-_{(n)}, J^{-L}_{(2)}] = \int d\Gamma_n \Phi_{(n)} \Big( \sum_{a=1}^n (J^{-L}_a)^t + \frac{1}{n} \sum_{b=1}^n p_b^- \partial_{p_a^R} \Big) p^-_{(n)} + \frac{1}{n} \Big( \sum_{a=1}^n \partial_{p_a^R} \sum_{b=1}^n p_b^- \Phi_{(n)} \Big) p^-_{(n)} \,, \qquad \text{(C.1)}$$

$$[J^{-L}_{(n)}, P^-_{(2)}] = \int d\Gamma_n \Phi_{(n)} \sum_{b=1}^n p_b^- \Big( j^{-L}_{(n)} + \frac{1}{n} \sum_{a=1}^n \frac{\lambda_a}{\beta_a} q^{-L}_{(n)} \Big) + \frac{1}{n} \Big( \sum_{a=1}^n \partial_{p_a^R} \sum_{b=1}^n p_b^- \Phi_{(n)} \Big) p^-_{(n)} \,, \qquad \text{(C.2)}$$

where notation $(J^{-L})^t$ is used for the operators obtainable from (2.36) by applying of transposition that is defined to be

$$\partial^t_{p^I} = -\partial_{p^I} \,, \qquad \theta^t = -\theta, \qquad (p^I)^t = p^I \,, \qquad \lambda^t = \lambda \,. \qquad \text{(C.3)}$$

Transposition on the product of bosonic ($B$) and fermionic ($F$) quantities is defined to be $(B_1B_2)^t = B_2^t B_1^t$, $(BF)^t = F^t B^t$, $(F_1F_2)^t = -F_2^t F_1^t$.

In cubic approximation the relations (C.1),(C.2) and commutator

$$[P^-_{(3)}, J^{-L}_{(2)}] = [J^{-L}_{(3)}, P^-_{(2)}] \qquad \text{(C.4)}$$

lead to the formula

$$\sum_{b=1}^3 p_b^- \Big( j^{-L}_{(3)} + \frac{1}{3} \sum_{a=1}^3 \frac{\lambda_a}{\beta_a} q^{-L}_{(3)} \Big) = \Big( \sum_{a=1}^3 (J_a^{-L})^t + \frac{1}{3} \sum_{b=1}^3 p_b^- \partial_{p_a^R} \Big) p^-_{(3)} \,. \qquad \text{(C.5)}$$

Taking into account that $p^-_{(3)}$ depends on $\mathbb{P}^I$ and $\Lambda$ and using $Li$ part of Eq.(4.4) we can cast an action of differential operator $\sum_{a=1}^3 (J_a^{-L})^t$ on $p^-_{(3)}$ into the form

$$\sum_{a=1}^3 (J_a^{-L})^t p^-_{(3)} = \Big( -\frac{\mathbb{P}^L}{3\hat{\beta}} \sum_{a=1}^3 \check{\beta}_a \beta_a \partial_{\beta_a} - \frac{1}{3} \sum_{a=1}^3 \frac{\lambda_a}{\beta_a} Q^{-L}(\Lambda) \Big) p^-_{(3)} \,. \qquad \text{(C.6)}$$

Plugging this relation into (C.5) and using the second relation in (4.46) and formula

$$\sum_{a=1}^3 \partial_{p_a^R} p^-_{(3)} = 0 \qquad \text{(C.7)}$$

we get from (C.5) the relation

$$\sum_{a=1}^3 p_a^- j^{-L}_{(3)} = -\frac{\mathbb{P}^L}{3\hat{\beta}} \sum_{a=1}^3 \check{\beta}_a \beta_a \partial_{\beta_a} p^-_{(3)} \,. \qquad \text{(C.8)}$$

Taking into account

$$\sum_{a=1}^3 p_a^- = \frac{|\mathbb{P}|^2}{2\hat{\beta}} \qquad \text{(C.9)}$$

we arrive at the relation (4.31) taken to be for transverse index $I = L$. Above-given calculations can extended to the cases of $j^{-R}_{(3)}$, $j^{-i}_{(3)}$ in a rather straightforward way. This leads to the formula given in (4.31).

# Appendix D Derivation of representation for 4-point vertex (5.29)

In this appendix we outline a procedure of solving the defining equations for 4-point vertex (5.18)-(5.21). First we express orbital momenta $L_{ab}^{IJ}$ in terms of the variables given in (5.5). The orbital momenta $L_{ab}^{Li}$ take then the form as in (4.16) and this allows us to write a solution of $Li$ parts of Eqs.(5.19) in the following form

$$p_{(4)}^{-} = E_{q_{13}} E_{q_{24}} V' \,, \tag{D.1}$$

$$V' \equiv V'(q_{_L}, \Lambda_{13}, \Lambda_{24}, \mathbb{P}_{13}^L, \mathbb{P}_{24}^L, \rho_{13}, \rho_{24}, \beta_a) \,, \tag{D.2}$$

where $E_{q_{ab}}$ and $q_{_L}^i$ are defined in (5.30),(5.32). Now we have to reformulate remaining equations in terms of $V'$. Moving the operators $J^{RL}$ $J^{ij}$ and $J^{Ri}$ throughout the operators $E_{q_{ab}}$ one can make sure that Eqs.(5.19) lead to the following equations for $V'$

$$(q_{_L}\partial_{q_{_L}} + 2\rho_{13}\partial_{\rho_{13}} + 2\rho_{24}\partial_{\rho_{24}} - \mathbb{P}_{13}^L\partial_{\mathbb{P}_{13}^L} - \mathbb{P}_{24}^L\partial_{\mathbb{P}_{24}^L} + M_{13}^{RL} + M_{24}^{RL})V' = 0 \,, \tag{D.3}$$

$$(q_{_L}^i\partial_{q_{_L}^j} - q_{_L}^j\partial_{q_{_L}^i} + M_{13}^{ij} + M_{24}^{ij})V' = 0 \,, \tag{D.4}$$

$$\begin{aligned} & \frac{q_{_L}^i}{2}(2\rho_{13}\partial_{\rho_{13}} - 2\rho_{24}\partial_{\rho_{24}} - \mathbb{P}_{13}^L\partial_{\mathbb{P}_{13}^L} + \mathbb{P}_{24}^L\partial_{\mathbb{P}_{24}^L} + M_{13}^{RL} - M_{24}^{RL})V' \\ & + \Big((\rho_{13}-\rho_{24})\partial_{q_{_L}^i} + M_{13}^{Ri} + M_{24}^{Ri} - \rho_{13}M_{13}^{Li} - \rho_{24}M_{24}^{Li} + \frac{q_{_L}^j}{2}(M_{13}^{ij} - M_{24}^{ij})\Big)V' = 0 \,. \end{aligned} \tag{D.5}$$

Doing the same for supercharges we get from (5.20) the equations

$$\Big(\frac{1}{\sqrt{2}}(\theta_{\Lambda_{13}}\not{\mathbb{P}}_{13} + \theta_{\Lambda_{24}}\not{\mathbb{P}}_{24}) + \frac{\rho_{13}\mathbb{P}_{13}^L\Lambda_{13}}{\beta_1\beta_3\beta_{24}} + \frac{\rho_{24}\mathbb{P}_{24}^L\Lambda_{24}}{\beta_2\beta_4\beta_{13}}\Big)V' = 0 \,, \tag{D.6}$$

$$(\mathbb{P}_{13}^L\theta_{\Lambda_{13}} + \mathbb{P}_{24}^L\theta_{\Lambda_{24}})V' = 0 \,. \tag{D.7}$$

Thus the generic equations (5.19),(5.20) are reduced to (D.4)-(D.7). We note that because Eqs.(5.20) are valid on the energy surface (5.13) we should also reduce our equations to the energy surface, *i.e.* we should express interaction vertex in terms of Mandelstam variable $u$ instead of $\rho_{13}$, $\rho_{24}$. Before doing that we find solution to the simple equation (D.7)

$$V' = V''(q_{_L}, \Lambda^L, \mathbb{P}_{13}^L, \mathbb{P}_{24}^L, \rho_{13}, \rho_{24}, \beta_a) \,, \tag{D.8}$$

where $\Lambda^L$ is given by (5.33). Taking into account that on the energy surface the variables $\rho_{13}$ and $\rho_{24}$ are expressible in terms of Mandelstam variable $u$ (5.13) we introduce the vertex $V'''$ by relation

$$V'' = V'''(q_{_L}, \Lambda^L, \mathbb{P}_{13}^L, \mathbb{P}_{24}^L, u, \beta_a) \,. \tag{D.9}$$

Now we have to repeat our procedure and rewrite remaining Eqs.(D.3)-(D.6) in terms of $V'''$. First let us consider Eq.(D.6). In terms of $V'''$ the equation (D.6) takes the form

$$\Big(\frac{1}{\sqrt{2}}\mathbb{P}_{13}^L\mathbb{P}_{24}^L\theta_{\Lambda^L}\not{q} + \frac{u\Lambda^L}{2\beta_{24}\mathbb{P}_{13}^L\mathbb{P}_{24}^L}\Big)V''' = 0 \,, \tag{D.10}$$

whose solution is easily found to be

$$V''' = E_u V^{\rm iv}(q_{_L}, \mathbb{P}^L_{13}, \mathbb{P}^L_{24}, u, \beta_a)\,. \tag{D.11}$$

where the operator $E_u$ is given in (5.31).

Next step is to consider Eqs.(D.3),(D.5). It is straightforward to demonstrate that in terms of $V^{\rm iv}$ these equations take the form

$$(q_{_L}\partial_{q_{_L}} - \mathbb{P}^L_{13}\partial_{\mathbb{P}^L_{13}} - \mathbb{P}^L_{24}\partial_{\mathbb{P}^L_{24}} + 4)V^{\rm iv} = 0\,, \tag{D.12}$$

$$\Big(\frac{q^i_{_L}}{2}(-\mathbb{P}^L_{13}\partial_{\mathbb{P}^L_{13}} + \mathbb{P}^L_{24}\partial_{\mathbb{P}^L_{24}}) + u\frac{(\beta_1\beta_3\mathbb{P}^{L2}_{24} - \beta_2\beta_4\mathbb{P}^{L2}_{13})}{2(\mathbb{P}^L_{13}\mathbb{P}^L_{24})^2}(\partial_{q^i_{_L}} - \frac{4q^i_{_L}}{q^2_{_L}})\Big)V^{\rm iv} = 0\,. \tag{D.13}$$

In addition to these equations there are two equations obtainable from Eqs.(5.18),(5.21)

$$(\mathbb{P}^L_{13}\partial_{\mathbb{P}^L_{13}} + \mathbb{P}^L_{24}\partial_{\mathbb{P}^L_{24}} + \beta_{13}\partial_{\beta_{13}} + y_{13}\partial_{y_{13}} + y_{24}\partial_{y_{24}} - 4)V^{\rm iv} = 0\,, \tag{D.14}$$

$$\Big(\frac{\mathbb{P}^L_{13}}{\beta_{13}}\partial_{y_{13}} + \frac{\mathbb{P}^L_{24}}{\beta_{24}}\partial_{y_{24}}\Big)V^{\rm iv} = 0\,, \tag{D.15}$$

where we use the notation

$$y_{13} \equiv \beta_1 - \beta_3\,, \qquad y_{24} \equiv \beta_2 - \beta_4\,. \tag{D.16}$$

In Eqs.(D.12)-(D.15) and below the vertex $V^{\rm iv}$ is considered to be function depending on three independent light cone momenta $\beta_{13}$, $y_{13}$, $y_{24}$ instead of four momenta $\beta_a$ subject to the conservation low $\sum_{a=1}^4 \beta_a = 0$. Eq.(D.14) is obtainable from Eq.(5.18), while Eq.(D.15) is obtainable from commutator $[P^-_{(4)}, J^{-L}_{(2)}] = 0$. Helpful relation to analyze this commutator is given in (C.1). Now we focus on the equations (D.12)-(D.15).

Solution to (D.15) is easily fixed to be

$$V^{\rm iv} = V^{\rm v}(u, q_{_L}, \mathbb{P}^L_{13}, \mathbb{P}^L_{24}, Y, \beta_{13})\,, \qquad Y \equiv \mathbb{P}^L_{13}y_{24} + \mathbb{P}^L_{24}y_{13}\,. \tag{D.17}$$

Plugging this representation into (D.13) we get the following equation

$$\begin{aligned}&\Big(\frac{q^i_{_L}}{2}(-\mathbb{P}^L_{13}\partial_{\mathbb{P}^L_{13}} + \mathbb{P}^L_{24}\partial_{\mathbb{P}^L_{24}}) + u\beta_{13}^2\frac{(\mathbb{P}^{L2}_{24} - \mathbb{P}^{L2}_{13})}{8(\mathbb{P}^L_{13}\mathbb{P}^L_{24})^2}(\partial_{q^i_{_L}} - \frac{4q^i_{_L}}{q^2_{_L}})\Big)V^{\rm v}\\ &-y^L\Big(-\frac{q^i_{_L}}{2}\partial_Y + \frac{uY}{8(\mathbb{P}^L_{13}\mathbb{P}^L_{24})^2}(\partial_{q^i_{_L}} - \frac{4q^i_{_L}}{q^2_{_L}})\Big)V^{\rm v} = 0\,,\end{aligned} \tag{D.18}$$

where

$$y^L \equiv y_{13}\mathbb{P}^L_{24} - y_{24}\mathbb{P}^L_{13}\,. \tag{D.19}$$

Because the vertex $V^{\rm v}$ does not depend on $y^L$ equation (D.15) implies that this vertex should satisfy the following two equations

$$\Big(\frac{q^i_{_L}}{2}(-\mathbb{P}^L_{13}\partial_{\mathbb{P}^L_{13}} + \mathbb{P}^L_{24}\partial_{\mathbb{P}^L_{24}}) + u\beta_{13}^2\frac{(\mathbb{P}^{L2}_{24} - \mathbb{P}^{L2}_{13})}{8(\mathbb{P}^L_{13}\mathbb{P}^L_{24})^2}(\partial_{q^i_{_L}} - \frac{4q^i_{_L}}{q^2_{_L}})\Big)V^{\rm v} = 0\,, \tag{D.20}$$

$$\Big(-\frac{q^i_{_L}}{2}\partial_Y + \frac{uY}{8(\mathbb{P}^L_{13}\mathbb{P}^L_{24})^2}(\partial_{q^i_{_L}} - \frac{4q^i_{_L}}{q^2_{_L}})\Big)V^{\rm v} = 0\,. \tag{D.21}$$

Solution to these equations is found to be[43]

$$V^{\rm v} = (q_{_L}^2)^2 V^{\rm vi}(u,\, v,\, \omega,\, \beta_{13})\,, \qquad \omega \equiv \mathbb{P}_{13}^L \mathbb{P}_{24}^L\,, \qquad v \equiv \frac{t-s}{2}\,, \tag{D.22}$$

where we use the following helpful relation for the variables $u$, $v$ and $Y$:

$$v = \frac{\mathbb{P}_{13}^L \mathbb{P}_{24}^L}{\beta_{13}^2} q_{_L}^2 + \frac{Y^2 u}{4\beta_{13}^2 \mathbb{P}_{13}^L \mathbb{P}_{24}^L} - \frac{\mathbb{P}_{13}^{L2} + \mathbb{P}_{24}^{L2}}{4\mathbb{P}_{13}^L \mathbb{P}_{24}^L} u\,. \tag{D.23}$$

Formula (D.22) implies the following representation for $V^{\rm iv}$ (see (D.17)):

$$V^{\rm iv} = (q_{_L}^2)^2 V^{\rm vi}(u,\, v,\, \omega,\, \beta_{13})\,. \tag{D.24}$$

Plugging the $V^{\rm iv}$ (D.24) into (D.12) we get solution for $V^{\rm vi}$:

$$V^{\rm vi}(u, v, \omega, \beta_{13}) = \omega^4 V^{\rm vii}(u, v, \beta_{13})\,, \tag{D.25}$$

which implies

$$V^{\rm iv} = \omega^4 (q_{_L}^2)^2 V^{\rm vii}(u,\, v,\, \beta_{13})\,. \tag{D.26}$$

Plugging (D.26) into (D.14) we get solution for vertex $V^{\rm vii}$:

$$V^{\rm vii}(u,\, v,\, \beta_{13}) = \frac{1}{\beta_{13}^4} V^{\rm viii}(u,\, v)\,. \tag{D.27}$$

For the case of supergravity theory the vertex $V^{\rm viii}$ should be symmetric with respect to Mandelstam variables $s$, $t$, $u$. To respect this requirement we take into account the relation (5.15) which implies that dependence on $u$, $v$ can be replaced by dependence on $s$, $t$, $u$ and therefore we can simply rewrite the vertex $V^{\rm viii}$ in the form $V^{\rm viii} = g(s,t,u)$, where the function $g(s,t,u)$ is considered to be symmetric in $s$, $t$, $u$.[44] Collecting all steps of derivation we get the following solution to $V^{\rm iv}$:

$$V^{\rm iv} = \frac{(\mathbb{P}_{13}^L \mathbb{P}_{24}^L)^4}{\beta_{13}^4} (q_{_L}^2)^2 g(s,t,u)\,. \tag{D.28}$$

The dependence on Mandelstam variables $s$, $t$, $u$ cannot be defined by commutation relations Poincaré superalgebra and this is freedom of our solution. Taking into account formulas (D.1),(D.8),(D.9),(D.11), (D.28) we arrive at formula (5.29).

# Appendix E $R^4$ and $F^4$ terms in light cone basis

In this appendix we explain how various $R^4$ (and $F^4$) terms given in (5.40)-(5.45) (and (7.25)) can be cast into light cone basis. This will allow us to relate normalization of

[43] Instead of $v$ (D.22) we could use another function of the Mandelstam variables, which is not constant on the surface $u = const$. We prefer to exploit the variable $v$ as this variable has simple transformation rule upon cyclic permutation of four external line indices 1,2,3,4. Namely, upon cyclic permutations of 1,2,3,4 we get $v \to -v$.

[44] Note that for the case of YM theories the vertex should be symmetric only upon cyclic permutations of 1,2,3,4. This is reason why for the case of YM theory we use representation $V^{\rm viii} = g(s,t)$ and impose constraint (7.21) in (see (7.20)).

covariant Lagrangian and corresponding supersymmetric light cone Hamiltonian. We begin our discussion with the gravitational $R^4$ terms.

Making Fourier transformation to momentum space for all coordinates except for the time $x^+$

$$\Phi(x) = \int \frac{d^{d-1}p}{(2\pi)^{(d-1)/2}} e^{\mathrm{i}(x^-\beta + x^I p^I)} \Phi(x^+, p) \tag{E.1}$$

(by setting $d = 11$ for fields of $11d$ supergravity) we cast covariant action corresponding to 4-point Lagrangian (5.37) into the form

$$\int d^{11}x \, \mathcal{L}_4(x) = \int dx^+ P^-_{(4)} \,, \tag{E.2}$$

where we introduce an appropriate 4-point Hamiltonian

$$P^-_{(4)} = \int d\Gamma_4(p) \mathcal{L}_4(p) \,, \tag{E.3}$$

and $\mathcal{L}_4$ indicates 4-point approximation of covariant Lagrangian (5.37). The measure $d\Gamma_4(p)$ is given by formula (3.8) in which we set $n = 4$. In what follows we assume: 1) massless fields are on mass-shell $(\mathrm{i}\partial_{x^+} + p^-)\Phi(x^+, p) = 0$, where $p^-$ is given in (2.34); 2) momenta of fields are restricted to the energy surface (5.11). To find $\mathcal{L}_4(p)$ we should find expressions for $W_1$ and $W_2$ in the 4-point approximation, which we shall denote as $W_1(p)$ and $W_2(p)$ (see (5.37))[45]

$$\mathcal{L}_4(p) = \kappa_{_{(4)}} W_{R^4}(p) \,, \qquad W_{R^4}(p) = W_1(p) + \frac{1}{16} W_2(p) \,. \tag{E.4}$$

We start our discussion with $W_1(p)$ term that is obtainable from (5.38).

Following Fourier transform (E.1) we introduce Fourier modes for linearized Riemann tensor and Lorentz connection

$$R_{\mu\nu}(p) \equiv p_\mu \omega_\nu(p) - p_\nu \omega_\mu(p) \,, \tag{E.5}$$

$$\omega^{\mu\, AB}(p) \equiv -p^A \bar{h}^{B\mu} + p^B \bar{h}^{A\mu} \,, \qquad \bar{h}^{A\mu} \equiv \sqrt{2}\,\kappa h^{A\mu} \,, \tag{E.6}$$

where $h^{A\mu} \equiv \delta^A_\nu h^{\nu\mu}$ and we keep a dependence on the gravitational constant $\kappa$ to respect expansion (4.40). Indices $A, B = 0, 1, \ldots, 10$ are flat Lorentz indices. In (E.5) and below the quantities $R_{\mu\nu}$ and $\omega^\mu$ stand for respective matrices $R^{AB}_{\mu\nu}$ and $\omega^{\mu\, AB}$. Taking into account (5.38) and representation for Riemann tensor (E.5) we get the following expression for $W_1(p)$:

$$\begin{aligned} W_1(p) \;\; &= \;\; \mathrm{Tr}\, -\frac{ut}{2}\omega_{12}\omega_{34} - \frac{st}{4}\omega_1^\mu \omega_2^\nu \omega_3^\mu \omega_4^\nu + 2t\omega_{12} b_{13} b_{24} + 2u\omega_{12} b_{23} b_{14} \\ &+ \;\; t\omega_1^\mu b_{12} \omega_3^\mu b_{34} + s\omega_1^\mu b_{32} \omega_3^\mu b_{14} \,, \end{aligned} \tag{E.7}$$

where we use the notation

$$\omega_{ab} \equiv \omega_a^\mu \omega_b^\mu \,, \qquad b_{ab} \equiv p_a^\mu \omega_b^\mu \,, \qquad \omega_a^\mu \equiv \omega^\mu(p_a) \,. \tag{E.8}$$

[45]Note that beyond of establishing relation (5.36) we confirmed ourselves that gravitational body of our Hamiltonian (5.1) is indeed related with Lagrangian (5.37) in which the coefficients in front of various $R^4$ terms (5.40)-(5.45) should be equal to those of Eq.(5.37). To keep discussion from becoming unwieldy here we do not discuss these relative coefficients in front of various $R^4$ terms and use the coefficients that are evident from Eq.(5.37).

$W_1(p)$ in (E.7) can be easily cast into light cone basis by noticing that in light cone gauge (4.42) we have the relations

$$\omega^+(p) = 0\,, \qquad \omega^-(p) = -\frac{p^I}{\beta}\omega^I(p)\,, \tag{E.9}$$

which lead to helpful formula

$$p_a^\mu \omega_b^\mu = \frac{1}{\beta_b}\mathbb{P}_{ab}^I \omega_b^I\,. \tag{E.10}$$

Making use of this formula in (E.7) gives representation for $W_1(p)$ in light cone basis

$$\begin{aligned} W_1(p) &= \mathrm{Tr}\, -\frac{ut}{2}\omega_{12}\omega_{34} - \frac{st}{4}\omega_1^I\omega_2^J\omega_3^I\omega_4^J \\ &+ \frac{2}{\beta_3\beta_4}(t\mathbb{P}_{13}^I\mathbb{P}_{24}^J + u\mathbb{P}_{23}^I\mathbb{P}_{14}^J)\omega_{12}\omega_3^I\omega_4^J \\ &+ \frac{1}{\beta_2\beta_4}(t\mathbb{P}_{12}^I\mathbb{P}_{34}^J - s\mathbb{P}_{23}^I\mathbb{P}_{14}^J)\omega_1^M\omega_2^I\omega_3^M\omega_4^J\,. \end{aligned} \tag{E.11}$$

We note that because of the first relation given in (E.9) the Lorentz invariant scalar products $\omega_{ab}$ are reduced to transverse scalar products $\omega_{ab} = \omega_a^I\omega_b^I$.

Now we turn to $W_2$. Transformation of $W_2$ to the light cone basis is simplified by using the relation

$$R_{\mu\nu}^{A_1B_1}(p_1)R_{\mu\nu}^{A_2B_2}(p_2) = -\frac{g^{IJ}(\mathbb{P}_{12})}{\beta_1\beta_2}\omega^{I\,A_1B_1}(p_1)\omega^{J\,A_2B_2}(p_2)\,, \tag{E.12}$$

where we use the notation

$$g^{IJ}(x) \equiv |x|^2\delta^{IJ} - 2x^Ix^J\,, \qquad |x|^2 \equiv x^Ix^I\,. \tag{E.13}$$

Formula (E.12) can be proved by using (E.5),(E.10) and representation for Mandelstam variables given in (5.16). Making use of formula (E.12) all $R^4$- terms in $W_2$ (5.42)-(5.45) can be cast into light cone basis in a rather straightforward way

$$R_{43}(p) = \frac{g^{IJ}(\mathbb{P}_{12})g^{MN}(\mathbb{P}_{34})}{\beta_1\ldots\beta_4}\mathrm{Tr}\,\omega_1^I\omega_3^J\,\mathrm{Tr}\,\omega_2^M\omega_4^N\,, \tag{E.14}$$

$$R_{44}(p) = \frac{g^{IJ}(\mathbb{P}_{12})g^{MN}(\mathbb{P}_{34})}{\beta_1\ldots\beta_4}\mathrm{Tr}\,\omega_1^I\omega_2^J\,\mathrm{Tr}\,\omega_3^M\omega_4^N\,, \tag{E.15}$$

$$R_{45}(p) = \frac{g^{IJ}(\mathbb{P}_{12})g^{MN}(\mathbb{P}_{34})}{\beta_1\ldots\beta_4}\mathrm{Tr}\,\omega_1^I\omega_2^J\omega_3^M\omega_4^N\,, \tag{E.16}$$

$$R_{46}(p) = \frac{g^{IJ}(\mathbb{P}_{13})g^{MN}(\mathbb{P}_{24})}{\beta_1\ldots\beta_4}\mathrm{Tr}\,\omega_1^I\omega_2^M\omega_3^J\omega_4^N\,. \tag{E.17}$$

Note that Tr in formulas (E.11),(E.14)-(E.17) still indicates trace over all Lorentz indices $A, B = 0, 1, \ldots, 10$. In fact formulas (E.11),(E.14)-(E.17) give a desired representation of

$R^4$ terms in the light cone basis. These representations can be reformulated in terms of $h^{\mu\nu}$ (E.6) by using the following helpful formulas

$$\mathrm{Tr}\,(\omega_a^\mu \omega_b^\nu) = \frac{h_a^{\mu I} g^{IJ}(\mathbb{P}_{ab}) h_b^{J\nu}}{\beta_a \beta_b}\,, \qquad h_a^{\mu\nu} \equiv h^{\mu\nu}(p_a)\,, \tag{E.18}$$

$$\omega_1^{\mu AC}\omega_2^{\nu CB} = -p_{12} h_1^{\mu A} h_2^{\nu B} - p_1^A p_2^B h_1^{\mu\rho} h_2^{\rho\nu} + h_1^{\mu A}\mathbb{P}_{12}^I h_2^{I\nu}\frac{p_2^B}{\beta_2} - \mathbb{P}_{12}^I h_1^{I\mu}\frac{p_1^A}{\beta_1} h_2^{\nu B}\,, \tag{E.19}$$

where $p_{12} \equiv p_1^\mu p_2^\mu$.

Now we would like to demonstrate how these results can be used to relate Lagrangian in covariant and light cone bases. It turns out that it suffices to consider terms proportional to $h_1^{LL} h_2^{RR} h_3^{RR} h_4^{RR}$. To this end we evaluate contribution of such terms to $W_1$ and $W_2$. We introduce

$$W_1(p)|, W_2(p)| = \frac{\bar{h}_1^{LL}\bar{h}_2^{RR}\bar{h}_3^{RR}\bar{h}_4^{RR}}{(\beta_1\ldots\beta_4)^2}\widetilde{W}_1(p), \widetilde{W}_2(p), \tag{E.20}$$

where notation $W_1(p)|$, $W_2(p)|$ indicates that we keep only those parts of $W_1(p)$, $W_2(p)$ which are proportional to $h_1^{LL} h_2^{RR} h_3^{RR} h_4^{RR}$. Making use of formulas (5.39), (E.11), (E.14)-(E.17) we get

$$\widetilde{W}_1(p) = \beta_1^3 su \mathbb{P}_{14}^L \mathbb{P}_{23}^L \Big( -2\beta_1 \mathbb{P}_{14}^L \mathbb{P}_{23}^L - \beta_3 (\mathbb{P}_{24}^L)^2 + \beta_2 (\mathbb{P}_{34}^L)^2 \Big)\,, \tag{E.21}$$

$$\widetilde{W}_2(p) = 16\beta_1^3 su \mathbb{P}_{14}^L \mathbb{P}_{23}^L \Big( \beta_3 (\mathbb{P}_{24}^L)^2 - \beta_2 (\mathbb{P}_{34}^L)^2 \Big)\,. \tag{E.22}$$

Taking into account the second formula in (5.37) we get

$$W_{R^4}(p) = \frac{\bar{h}_1^{LL}\bar{h}_2^{RR}\bar{h}_3^{RR}\bar{h}_4^{RR}}{(\beta_1\ldots\beta_4)^2}\Big( -2\beta_1^4 su (\mathbb{P}_{14}^L \mathbb{P}_{23}^L)^2 \Big)\,. \tag{E.23}$$

To relate this result to the light cone Hamiltonian (5.1),(5.29) we should find contribution of $h_1^{LL} h_2^{RR} h_3^{RR} h_4^{RR}$- terms to the Hamiltonian (5.1). To this end we note the relation

$$\Phi_1|_{h_1^{LL}}\,\Phi_2\Phi_3\Phi_4|_{h_2^{RR}h_3^{RR}h_4^{RR}} = \frac{\beta_1^4(\epsilon\lambda_2^8)(\epsilon\lambda_3^8)(\epsilon\lambda_4^8)}{4(\beta_1\ldots\beta_4)^2} h_1^{LL} h_2^{RR} h_3^{RR} h_4^{RR} \tag{E.24}$$

and this implies the following relation

$$\int d\Gamma_4(\lambda)\prod_{a=1}^4 \Phi(p_a,\lambda_a)\, p_{(4)}^-| = g_0 \frac{h_1^{LL} h_2^{RR} h_3^{RR} h_4^{RR}}{(\beta_1\ldots\beta_4)^2}\,\beta_1^4\,\frac{(\mathbb{P}_{13}^L \mathbb{P}_{24}^L)^4}{\beta_{13}^4}(q_{_L}^2)^2\,. \tag{E.25}$$

where the measure $d\Gamma_4(\lambda)$ is given by formula (3.9) in which we set $n = 4$. Making use then the formula (5.52) we get the following contribution of $h_1^{LL} h_2^{RR} h_3^{RR} h_4^{RR}$ terms to the 4-point Hamiltonian

$$P_{(4)}^-| = g_0 \int d\Gamma_4(p)\,\frac{h_1^{LL} h_2^{RR} h_3^{RR} h_4^{RR}}{(\beta_1\ldots\beta_4)^2}\,(-3su\beta_1^4\,(\mathbb{P}_{14}^L \mathbb{P}_{23}^L)^2)\,. \tag{E.26}$$

Comparing this formula with (E.4),(E.23)(see also the second formula in (E.6)) we arrive at (5.36).

The analysis above-given is extended to the case of higher derivative Lagrangian (5.46) straightforwardly because the factor $f(s,t,u)$ is symmetric with respect to Mandelstam variables and therefore this factor does not affect derivation.

These considerations can be easily extended to the case of $10d$ SYM theory. In this case the covariant Lagrangian $\mathcal{L}_{_{F^4}}$ given by (7.25) leads to the 4-point Hamiltonian

$$P^-_{(4)} = \int d\Gamma_4(p)\,\mathcal{L}_4(p)\,, \qquad \mathcal{L}_4(p) = g_{_{F^4}} W_{_{F^4}}(p)\,, \tag{E.27}$$

where the measure $d\Gamma_4(p)$ is given by formula (3.8) in which we set $n=4$, $d=10$. It easy to see that if in expression for $W_1$ (5.38) we replace the Lorentz connection $\omega_\mu$ by the gauge field $\phi_\mu$ then we get $W_{F^4}$ (7.26). Therefore Lorentz covariant representation for $W_{F^4}(p)$ in terms of gauge field $\phi^\mu$ can be obtained from (E.7) by making there just mentioned replacement

$$\begin{aligned} W_{_{F^4}}(p) &= \mathrm{Tr}\, -\frac{ut}{2}\phi_{12}\phi_{34} - \frac{st}{4}\phi_1^\mu\phi_2^\nu\phi_3^\mu\phi_4^\nu + 2t\phi_{12}b_{13}b_{24} + 2u\phi_{12}b_{23}b_{14} \\ &+ t\phi_1^\mu b_{12}\phi_3^\mu b_{34} + s\phi_1^\mu b_{32}\phi_3^\mu b_{14}\,, \end{aligned} \tag{E.28}$$

where we use the notation

$$\phi_{ab} \equiv \phi_a^\mu\phi_b^\mu\,, \qquad b_{ab} \equiv p_a^\mu\phi_b^\mu\,, \qquad \phi_a^\mu \equiv \phi^\mu(p_a)\,. \tag{E.29}$$

Making use of light cone gauge and the relations similar to the ones in (E.9),(E.10) adopted for gauge fields we get representation $W_{F^4}(p)$ in the light cone basis (cf. (E.11)):

$$\begin{aligned} W_{_{F^4}}(p) &= \mathrm{Tr}\, -\frac{ut}{2}\phi_{12}\phi_{34} - \frac{st}{4}\phi_1^I\phi_2^J\phi_3^I\phi_4^J \\ &+ \frac{2}{\beta_3\beta_4}(t\mathbb{P}_{13}^I\mathbb{P}_{24}^J + u\mathbb{P}_{23}^I\mathbb{P}_{14}^J)\phi_{12}\phi_3^I\phi_4^J \\ &+ \frac{1}{\beta_2\beta_4}(t\mathbb{P}_{12}^I\mathbb{P}_{34}^J - s\mathbb{P}_{23}^I\mathbb{P}_{14}^J)\phi_1^M\phi_2^I\phi_3^M\phi_4^J\,. \end{aligned} \tag{E.30}$$

To fix normalization it suffices to analyze terms proportional to $\phi_1^L\phi_2^R\phi_3^R\phi_4^R$. On the one hand $W_{_{F^4}}(p)$ gives the following contribution to $\phi_1^L\phi_2^R\phi_3^R\phi_4^R$- term:

$$W_{_{F^4}}(p)|_{\phi_1^L\phi_2^R\phi_3^R\phi_4^R} = \mathrm{Tr}\,\frac{\phi_1^L\phi_2^R\phi_3^R\phi_4^R}{\beta_1\beta_2\beta_3\beta_4}\,\frac{(\mathbb{P}_{13}^L\mathbb{P}_{24}^L)^2}{\beta_{13}^2}q_{_L}^2(-2\beta_1^2)\,. \tag{E.31}$$

On the other hand our Hamiltonian (7.19) gives the following contribution to $\phi_1^{LL}\phi_2^{RR}\phi_3^{RR}\phi_4^{RR}$ - term:

$$P^-_{(4)}| = g_0\mathrm{Tr}\int d\Gamma_4(p)\,\frac{\phi_1^L\phi_2^R\phi_3^R\phi_4^R}{\beta_1\ldots\beta_4}\,\frac{(\mathbb{P}_{13}^L\mathbb{P}_{24}^L)^2}{\beta_{13}^2}q_{_L}^2(-4\beta_1^2)\,. \tag{E.32}$$

Comparison of this formula with Eqs.(E.27),(E.31) leads to the relationship for the coefficients $g_0$ and $g_{_{F^4}}$ given in Eq.(7.27) (see (7.22)). Generalization of this consideration to the case of the Lagrangian (7.28) to get the relationship (7.30) is straightforward.

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